Physics 5013. Homework 8
Due Wednesday, December 13, 2006

November 21, 2006

1. Suppose we have a second-order differential operator of the form

$$L = \frac{1}{f}\frac{d}{dx}\left(f\frac{d}{dx}\right) + q,$$

where f and q are functions of x. If y_1 and y_2 are independent solutions of

$$Ly = 0,$$

the Wronskian

$$\Delta(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is different from zero. Prove that

$$\frac{d}{dx}\Delta = -\Delta\frac{d}{dx}\ln f,$$

and that

$$y_2(x) = \Delta(x_0) f(x_0) y_1(x) \int_{x_0}^{x} \frac{du}{f(u) y_1^2(u)},$$

where x_0 is a point at which

$$\begin{aligned} y_2(x_0) &= 0, \quad y_1(x_0) \neq 0, \\ f(x_0) &\neq 0, \quad y_1'(x_0) \neq 0. \end{aligned}$$

2. Recall that the Bessel functions of integer order are defined by

$$e^{(x/2)(z-1/z)} = \sum_{m=-\infty}^{\infty} z^m J_m(x),$$

or, with $x = kr$, $z = ie^{i\phi}$.

$$e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr).$$

Use this expression in the *two-dimensional* completeness statement for the functions

$$\frac{1}{2\pi} e^{i\mathbf{k}\cdot\mathbf{r}},$$

that is,

$$\int \frac{(d\mathbf{k})}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{r}'} = \delta(\mathbf{r} - \mathbf{r}'),$$

where the right-hand side is a two-dimensional delta function, which in polar coordinates is

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r}\delta(r - r')\delta(\theta - \theta'),$$

and $(d\mathbf{k})$ is the two-dimensional integration element, which is correspondingly given in polar coordinates as

$$(d\mathbf{k}) = k\,dk\,d\alpha.$$

In this way derive the completeness property of the Bessel functions,

$$\int_0^\infty k\,dk\,J_m(kr)J_m(kr') = \frac{1}{r}\delta(r - r').$$

3. Determine, directly, the one-dimensional Green's function $G_k(r, r')$ for the Bessel differential operator of order zero; that is, solve

$$\frac{d}{dr}\left(r\frac{dG_k}{dr}\right) + k^2 r G_k = \delta(r - r'), \quad 0 \le r \le a,$$

subject to the boundary condition

$$G_k(a, r') = 0.$$

Show that G_k is singular wherever $k = k_n$, where

$$J_0(k_n a) = 0.$$

From the behavior of G_k at this singularity determine the normalization integral

$$\int_0^a r J_0^2(k_n r)\, dr.$$

[Hint: It is necessary to use both the regular solution to Bessel's equation of order zero, J_0, and the irregular solution, N_0. The result of problem 1, as well as the asymptotic behaviors

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right), \quad x \gg 1,$$

$$N_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right), \quad x \gg 1,$$

will be helpful.]

4. Find the Green's function for the two-dimensional Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) G(x, y; x', y') = \delta(x - x')\delta(y - y')$$

in the interior of a square of side a, expressed as an eigenfunction expansion. With the origin of coordinates chosen to be one corner of the square, the boundary conditions are as follows (see Fig. 1):

$$\begin{aligned} G(0, y; x', y') &= 0, \\ G(a, y; x', y') &= 0, \\ \frac{\partial}{\partial y} G(x, 0; x', y') &= 0, \\ \frac{\partial}{\partial y} G(x, a; x', y') &= 0. \end{aligned}$$

5. (a) Find the Green's function for Laplace's equation,

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

in a three-dimensional region lying between the two planes $x = 0$ and $x = a$ as shown in Fig. 2 with the boundary conditions

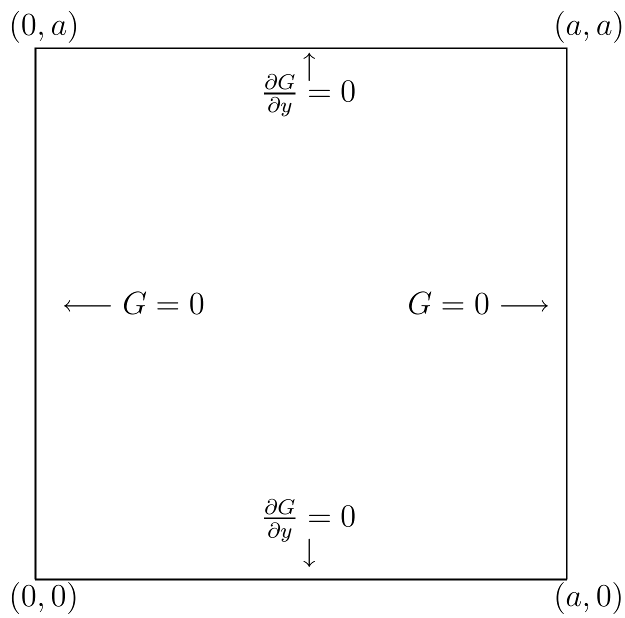

Figure 1: Boundary conditions for the Green's function in Problem 4.

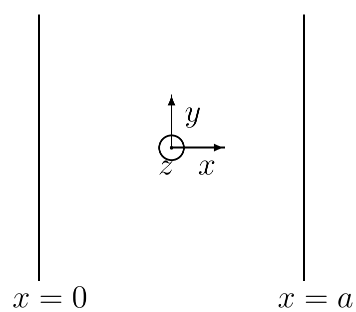

Figure 2: Coordinate system for Problem 5.

$$
\begin{aligned}
G(0,y,z;x',y',z') &= 0, \\
G(a,y,z;x',y',z') &= 0,
\end{aligned}
$$

using the following method: Show that G can be written in the form

$$
G(\mathbf{r},\mathbf{r}') = \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \, e^{ik_y(y-y')} e^{ik_z(z-z')} g_{k_\perp^2}(x,x'),
$$

where $k_\perp^2 = k_y^2 + k_z^2$, and find $g_{k_\perp^2}$ in closed form by using the discontinuity method.

(b) By examining the singularities of $g_{k_\perp^2}(x,x')$ with respect to $k_\perp^2$ find the normalized eigenfunctions and eigenvalues of d^2/dx^2 subject to the homogeneous Dirichlet boundary conditions at $x=0$ and $x=a$.

(c) Using the result of (b) and the generating function for the Bessel functions

$$
e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{m=-\infty}^{\infty} z^m J_m(x),
$$

prove that

$$
G(\mathbf{r},\mathbf{r}') = -\frac{1}{2\pi}\int_0^{\infty} dk \, k J_0(kR) \frac{2}{a} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a}}{k^2 + \left(\frac{n\pi}{z}\right)^2},
$$

where $R^2 = (y-y')^2 + (z-z')^2$.