Physics 5013. Homework 6
Due Wednesday, November 16, 2011

November 4, 2011

1. Evaluate the following divergent sums using both the Euler and the generic methods:

   (a)
   $$1 + 0 - 1 + 0 - 1 + 0 + 1 + 0 + 1 + 0 - 1 + \dots,$$

   (b)
   $$1 + 0 + 0 - 1 + 1 + 0 + 0 - 1 + 1 + 0 + 0 + \dots,$$

   (c)
   $$1 - 1 + 0 + 0 + 1 - 1 + 0 + 0 + \dots.$$

2. Evaluate the following series using both the Euler and the Borel methods:
$$\sum_{n=0}^{\infty}(-1)^n n^2.$$

3. Show that $0! - 2! + 4! - 6! + \dots$ is not Borel summable but that $0! + 0 - 2! + 0 + 4! + 0 - 6! + 0 + \dots$ is. Compute the Borel sum of the latter.

4. Consider the function
$$\frac{1}{z}\log(1+z).$$
Derive the $[3,3]$ Padé approximant stated in class
$$P_3^3(z) = \frac{1 + \frac{17}{14}z + \frac{1}{3}z^2 + \frac{1}{140}z^3}{1 + \frac{12}{7}z + \frac{6}{7}z^2 + \frac{4}{35}z^3}.$$
Similarly, work out $P_4^3(z)$, and verify the values given in class for $z = 0.5$, 1, and 2.

5. The Stirling series for the Gamma function is for $n \to \infty$,

$$\Gamma(n) = (n-1)! \sim \left(\frac{2\pi}{n}\right)^{1/2} \left(\frac{n}{e}\right)^n \left(1 + \frac{A_1}{n} + \frac{A_2}{n^2} + \frac{A_3}{n^3} + \frac{A_4}{n^4} + \dots\right),$$

where

$$\begin{aligned} A_1 &= \frac{1}{12}, \\ A_2 &= \frac{1}{288}, \\ A_3 &= -\frac{139}{51840}, \\ A_4 &= -\frac{571}{2488320}. \end{aligned}$$

Compute the $[1,1]$ and $[2,2]$ Padé approximants for $\Gamma(x)(e/x)^x\sqrt{x/2\pi}$. Compare numerically the values so obtained with the exact function for $x = 0.2$, 0.5, and 1.0, which are *small* values of x. Can a more accurate approximation be obtained by averaging P_1^1 and P_2^2?

6. Compute the first three terms of the continued-fraction coefficients of the series:

(a)
$$\sum_{n=0}^{\infty} (n!)^2(-z)^n,$$

(b)
$$\sum_{n=0}^{\infty} \frac{(-z)^n}{(2n)!},$$

(c)
$$\sum_{n=0}^{\infty} \frac{z^n}{n^2+1}.$$

7. Consider a continued-fraction representation of the exponential function in the form
$$e^x = \frac{c_0}{1 + \frac{c_1 z}{1 + \frac{c_2 z}{1 + \frac{c_3 z}{1 + \dots}}}}.$$

Show that

$$c_0 = -c_1 = 1, \quad c_{2n} = \frac{1}{4n-2}, \quad c_{2n+1} = -\frac{1}{4n+2}, \quad n \geq 1.$$

How many terms must be included to compute e to 8 significant figures?