Physics 5013. Homework 7
Due Monday, November 20, 2006

November 7, 2006

Problems in Bender and Orzag:

Chapter 6, pp. 313-4: **6.71, 6.76**

1. Show that

$$(2^s - 1)\zeta(s) = \frac{2^{s-1}s}{s-1} + 2\int_0^\infty \left(\frac{1}{4} + y^2\right)^{-s/2} \sin(s \arctan 2y) \frac{dy}{e^{2\pi y} - 1}.$$

2. One representation for the gamma function is

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i}\int_C e^t t^{-z}\, dt$$

where the contour of integration is as shown in Figure 1. Use the method of steepest descents to derive Stirling's approximation,

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}, \quad x \to +\infty.$$

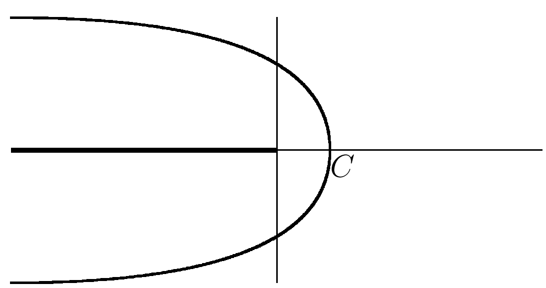

Figure 1: Contour C used for defining the Hankel representation of the gamma function.

3. The Airy function has the asymptotic expansion

$$\mathrm{Ai}(x) \sim \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-2x^{3/2}/3}[1 + O(x^{-3/2}) + O(x^{-3}) + \ldots].$$

   Calculate the $O(x^{-3/2})$ and the $O(x^{-3})$ terms.

4. Using the integral representation for the Hankel function of the first kind

$$H_\nu^{(1)}(z) = \frac{e^{-i\nu\pi/2}}{\pi}\int_{-\pi/2+i\infty}^{\pi/2-i\infty} d\phi\, e^{i(z\cos\phi+\nu\phi)},$$

   derive Debye's asymptotic expansion of $\tan\alpha > 0$ and ν large and positive:

$$H_\nu^{(1)}(\nu\sec\alpha) \sim \sqrt{\frac{2}{\pi\nu\tan\alpha}}e^{-i\pi/4}e^{i\nu(\tan\alpha-\alpha)}\left(1 + \frac{u_1(\cot\alpha)}{i\nu} + O(1/\nu^2)\right),$$

   where

$$u_1(t) = \frac{3t + 5t^3}{24}.$$