Physics 5013. Homework 6
Due Wednesday, October 18, 2006

October 9, 2006

1. By integrating $\frac{e^{\pm aiz}}{e^{2\pi z}-1}$ around a rectangle whose corners are 0, R, $R+i$, i (the rectangle being indented at 0 and i) and letting $R \to \infty$, show that
$$\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1} dx = \frac{1}{4}\frac{e^a+1}{e^a-1} - \frac{1}{2a},$$
a result due to Legendre.

2. Show that if $a > 0$, $b > 0$,
$$\int_0^\infty e^{a\cos bx}\sin(a\sin bx)\frac{dx}{x} = \frac{\pi}{2}(e^a - 1).$$

3. Show that
$$\int_0^{\pi/2} \frac{a\sin 2x}{1-2a\cos 2x + a^2} x\, dx = \begin{cases} \frac{\pi}{4}\log(1+a), & -1 < a < 1, \\ \frac{\pi}{4}\log(1+a^{-1}), & a^2 > 1. \end{cases}$$

4. Evaluate
$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}(1+x^2)^2}$$
by using a contour which encircles the branch line given in Problem 4.7, and closed by a circle at infinity. Equivalently, consider a contour of two parts: one that just encloses the branch line from $z = -1$ to $z = +1$ and another being a circle about the origin of very large radius. Between these two contours, the function
$$f(z) = \sqrt{1-z^2}$$
is analytic.

5. Recall the generating function defining the Bernoulli numbers:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Show that

$$B_n = \frac{n!}{2\pi i} \oint_{C_0} \frac{z}{e^z - 1} \frac{dz}{z^{n+1}},$$

where C_0 is a circle about the origin with radius $|z| < 2\pi$. From this integral find B_0, B_1 directly. By distorting C_0 into C, an infinite circle about the origin (and hence crossing an infinite number of poles!), show that for n even, $n \geq 2$,

$$B_n = -\frac{(-1)^{n/2} 2\, n!}{(2\pi)^n} \zeta(n),$$

where

$$\zeta(n) = \sum_{p=1}^{\infty} p^{-n}.$$

Use the residue theorem to evaluate the following integrals:

6.

$$\int_0^{2\pi} \frac{d\theta}{1 + a \sin^2 \theta}, \quad a > 0.$$

7.

$$\int_0^{\infty} \frac{dx \, \sin x}{x(x^2 + a^2)}, \quad a > 0.$$

8.

$$\int_0^{\infty} \frac{x^{2a-1}}{1 + x^2}, \quad 0 < a < 1.$$

9.

$$P \int_0^{\infty} dx \, \frac{\sqrt{x}}{x^2 - a^2}, \quad a > 0.$$