Physics 5013. Homework 4
Due Wednesday, October 12, 2011

October 1, 2011

1. Prove that the value of an analytic function at the center of a circle is equal to the mean of the values of the function on the circumference of the circle, provided that the function is analytic everywhere inside and on the circle. That is, if the circle has radius r,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) \, d\theta.$$

2. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a simply connected region R, the functions u and v do not attain local maxima or minima at any interior point of R. Prove this theorem two ways:

   (a) Use the Cauchy-Riemann conditions. [Assume, for example, that $\partial^2 u / \partial x^2 > 0$ at the stationary point.]

   (b) Use the Cauchy integral formula. [Take as contour a small circle about the stationary point. Then you may use the result of Problem 1.]

3. Let γ be a simple closed curve inside and on which $f(z)$ is analytic. Suppose that f does not vanish at any point on γ. Show that the integral

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz$$

equals the number of zeroes of f inside γ.

4. Show that, if $g(z)$ is analytic on and within a closed contour γ, and $f(z)$ has zeroes within γ, the zero at z_n having multiplicity α_n, $n = 1, 2, \ldots, N$, and poles within γ at z'_m, the order of the pole at z'_m being β_m, $m = 1, 2, \ldots, M$, α_n and β_m being integers, that

$$\frac{1}{2\pi i} \oint_\gamma dz\, g(z) \frac{d}{dz} \ln f(z) = \sum_{n=1}^N \alpha_n g(z_n) - \sum_{m=1}^M \beta_m g(z'_m).$$

5. An example of the use of this formula is in evaluating the zero-point energy of oscillation on a string which is tied down at points $x = 0$ and $x = a$. If the speed of the waves on the string is c, the sum of the zero-point oscillation energies on the string is

$$E_0 = \frac{1}{2} \sum \hbar\omega = \frac{\hbar c}{2} \sum_{n=1}^\infty \frac{n\pi}{a}.$$

This is divergent, yet may be evaluated as follows.

(a) Show that
$$E_0 = \sum \frac{\hbar c}{2} \int_C \frac{d\omega}{2\pi i} \omega \frac{d}{d\omega} \ln \sin \omega a.$$
Here C is a contour which encircles all the zeros of the sine function on the positive real axis.

(b) Now rotate formally $\omega \to i\zeta$, where ζ is now regarded as real. That is, we open up the contour around the poles on the real ω axis to one running along the imaginary axis. In doing so, omit a divergent term in the integrand proportional to ζa, and show that what is left is
$$E_0 = -\frac{\hbar c a}{\pi} \int_0^\infty d\zeta \zeta \frac{1}{e^{2\zeta a} - 1}.$$
(Where does the minus sign come from?)

(c) Evaluate this using the identity (prove this by expanding the denominator)
$$\int_0^\infty \frac{dx\, x^{n-1}}{e^x - 1} = \Gamma(n)\zeta(n),$$
in terms of the Riemann zeta function and the gamma function. You will need the value $\zeta(2) = \pi^2/6$. The result is called the Lüscher energy, the Casimir energy of interaction between two Dirichlet points, and is used in the quark model.

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6. The quantum vacuum energy (Casimir energy) of a scalar field living on the surface of the three-dimensional sphere of radius a is given by the exact formula for an arbitrary temperature T:

$$U = \frac{1}{240a} + \frac{1}{a}\sum_{n=1}^{\infty} \frac{n^3}{e^{n/aT} - 1}. \tag{1}$$

(a) Show that for low temperatures, $aT \ll 1$, this implies that

$$U \sim \frac{1}{a}\frac{1}{240}, \tag{2}$$

up to exponentially small corrections.

(b) Now use the Euler-Maclaurin sum formula to evaluate the sum in (1) and derive a formula valid for high temperatures,

$$U \sim \frac{(2\pi aT)^4}{240a}, \qquad aT \gg 1, \tag{3}$$

up to exponentially small corrections in the high temperature limit.

(c) Evaluate the latter by proving the Poisson sum formula from the identity

$$\sum_{n=-\infty}^{\infty} e^{-i2\pi nx} = \sum_{n=-\infty}^{\infty} \delta(x - n) \tag{4}$$

(why is this true?) from which one can deduce from the definition of the Fourier transform c of a function b,

$$c(\alpha) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-i\alpha x} b(x), \tag{5}$$

the equality

$$\sum_{n=-\infty}^{\infty} b(n) = 2\pi \sum_{n=-\infty}^{\infty} c(2\pi n). \tag{6}$$

Equation (6) is the Poisson sum formula.

(d) Show that the Fourier transform of

$$b(x) = \frac{1}{a}\begin{cases} \frac{x^3}{e^{x/aT}-1}, & x \geq 0, \\ 0, & x \leq 0 \end{cases} \tag{7}$$

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is

$$c(\alpha) = \frac{1}{2\pi a} \sum_{k=0}^{\infty} \frac{\Gamma(4)}{[(k+1)/aT + i\alpha]^4}, \tag{8}$$

and therefore show from (6) that (1) can be written in the form

$$U = \frac{1}{240a} + \frac{1}{a}(aT)^4\Gamma(4) \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{1}{[1+k-i2\pi aTn]^4}. \tag{9}$$

By interchanging the order of the two infinite summations here, show that this implies that the energy (1) can be written in the alternative form

$$U = \frac{(2\pi aT)^4}{240a} + \frac{1}{a}(2\pi aT)^4 \sum_{n=1}^{\infty} \frac{n^3}{e^{4\pi^2 naT} - 1}, \tag{10}$$

which indeed exhibits the exponentially small corrections to (3) for high temperature. Note that (1) and (10) are exactly equal, yet the first is especially adapted to describe the low temperature limit, and the second is useful for the high termperature limit.

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