

Physics 5013. Homework 4

Due Friday, October 6, 2006

September 25, 2006

1. Prove that the value of an analytic function at the center of a circle is equal to the mean of the values of the function on the circumference of the circle, provided that the function is analytic everywhere inside and on the circle. That is, if the circle has radius r ,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

2. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a simply connected region R , the functions u and v do not attain local maxima or minima at any interior point of R . Prove this theorem two ways:
 - (a) Use the Cauchy-Riemann conditions. [Assume, for example, that $\partial^2 u / \partial x^2 > 0$ at the stationary point.]
 - (b) Use the Cauchy integral formula. [Take as contour a small circle about the stationary point. Then you may use the result of Problem 1.]
3. Let γ be a simple closed curve inside and on which $f(z)$ is analytic. Suppose that f does not vanish at any point on γ . Show that the integral

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz$$

equals the number of zeroes of f inside γ .

4. The quantum vacuum energy (Casimir energy) of a scalar field living on the surface of the three-dimensional sphere of radius a is given by the exact formula for an arbitrary temperature T :

$$U = \frac{1}{240a} + \frac{1}{a} \sum_{n=1}^{\infty} \frac{n^3}{e^{n/aT} - 1}. \quad (1)$$

- (a) Show that for low temperatures, $aT \ll 1$, this implies that

$$U \sim \frac{1}{a} \frac{1}{240}, \quad (2)$$

up to exponentially small corrections.

- (b) Now use the Euler-Maclaurin sum formula to evaluate the sum in (1) and derive a formula valid for high temperatures,

$$U \sim \frac{(2\pi aT)^4}{240a}, \quad aT \gg 1, \quad (3)$$

up to exponentially small corrections in the high temperature limit.

- (c) Evaluate the latter by proving the Poisson sum formula from the identity

$$\sum_{n=-\infty}^{\infty} e^{-i2\pi nx} = \sum_{n=-\infty}^{\infty} \delta(x - n) \quad (4)$$

(why is this true?) from which one can deduce from the definition of the Fourier transform c of a function b ,

$$c(\alpha) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-i\alpha x} b(x), \quad (5)$$

the equality

$$\sum_{n=-\infty}^{\infty} b(n) = 2\pi \sum_{n=-\infty}^{\infty} c(2\pi n). \quad (6)$$

Equation (6) is the Poisson sum formula.

- (d) Show that the Fourier transform of

$$b(x) = \frac{1}{a} \begin{cases} \frac{x^3}{e^{x/aT} - 1}, & x \geq 0, \\ 0, & x \leq 0 \end{cases} \quad (7)$$

is

$$c(\alpha) = \frac{1}{2\pi a} \sum_{k=0}^{\infty} \frac{\Gamma(4)}{[(k+1)/aT + i\alpha]^4}, \quad (8)$$

and therefore show from (6) that (1) can be written in the form

$$U = \frac{1}{240a} + \frac{1}{a}(aT)^4 \Gamma(4) \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{1}{[1+k-i2\pi aTn]^4}. \quad (9)$$

By interchanging the order of the two infinite summations here, show that this implies that the energy (1) can be written in the alternative form

$$U = \frac{(2\pi aT)^4}{240a} + \frac{1}{a}(2\pi aT)^4 \sum_{n=1}^{\infty} \frac{n^3}{e^{4\pi^2 naT} - 1}, \quad (10)$$

which indeed exhibits the exponentially small corrections to (3) for high temperature. Note that (1) and (10) are exactly equal, yet the first is especially adapted to describe the low temperature limit, and the second is useful for the high temperature limit.

5. Show that the function $1/z^2$ represents the analytic continuation of the function defined by the series

$$\sum_{n=0}^{\infty} (n+1)(z+1)^n, \quad |z+1| < 1,$$

into the domain consisting of all points in the z plane except $z=0$.

6. Show that

$$e^{uz+v/z} = a_0 + a_1 z + a_2 z^2 + \dots + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots,$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_0^{2\pi} e^{(u+v)\cos\theta} \cos[(u-v)\sin\theta - n\theta] d\theta, \\ b_n &= \frac{1}{2\pi} \int_0^{2\pi} e^{(u+v)\cos\theta} \cos[(v-u)\sin\theta - n\theta] d\theta. \end{aligned}$$

7. Show that the function

$$f(z) = (z^2 - 1)^{1/2}$$

is single valued when it is defined with the branch line running along the real axis from -1 to $+1$. [Hint: Consider the net phase change in f when the branch line is encircled once.]