Physics 5013. Homework 2
Due Friday, September 17, 2010

September 10, 2010

1. Show that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

   is equal to $\frac{1}{(1-z)^2}$ when $|z| < 1$ and is equal to $\frac{1}{z(1-z)^2}$ when $|z| > 1$. Is this fact connected with the theory of uniform convergence?

2. Show that the series

$$2\sin\frac{1}{3z} + 4\sin\frac{1}{9z} + \ldots + 2^n \sin\frac{1}{3^n z} + \ldots$$

   converges absolutely for all values of z except for $z = 0$, but show that it does not converge uniformly near $z = 0$.

3. If (for x real)

$$u_n(x) = -2(n-1)^2 x e^{-(n-1)^2 x^2} + 2n^2 x e^{-n^2 x^2},$$

   show that $\sum_{n=1}^{\infty} u_n(x)$ does not converge uniformly near $x = 0$.

4. For what range of positive values of x is

$$\sum_{n=0}^{\infty} \frac{1}{1+x^n}$$

   (a) Convergent?

(b) Uniformly convergent?

5. In numerical analysis it is often convenient to replace derivatives by finite differences. For example, we might represent the second derivative of a function as follows:

$$\frac{d^2}{dx^2}\psi(x) \approx \frac{1}{h^2}[\psi(x+h) - 2\psi(x) + \psi(x-h)].$$

Regarding h as a small parameter, find the error in this approximation.

6. Compute e from the Taylor series of e^x about 0 to 16 significant figures.

7. Given that

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \frac{\pi}{4},$$

expand the integrand into a series and integrate term by term, obtaining

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots + (-1)^n \frac{1}{2n+1} + \cdots,$$

which is Leibnitz's formula for π (actually discovered by Gregory in 1671). This formula converges so slowly that it is quite useless for numerical work: compute the first 100 terms and see for yourself!