Physics 5013. Homework 7
Due Friday, December 2, 2011

November 30, 2011

1. An integral representation for the modified Bessel function $K_\nu(x)$ is

$$K_\nu(x) = \frac{1}{2}\int_{-\infty}^{\infty} dt\, e^{-x\cosh t + \nu t}.$$

Show that

$$K_{ip}(x) = \sqrt{2\pi}(p^2 - x^2)^{-1/4} e^{-p\pi/2} \sin\phi(x).$$

where

$$\phi(x) - p\cosh^{-1}(p/x) + \sqrt{p^2 - x^2} \sim \frac{\pi}{4}, \quad x \to +\infty, \quad p/x \to +\infty.$$

Hint: The contribution comes from the neighborhood of two saddle points satisfying $\sinh t = ip/x$. Explain why it is that although there are an infinite number of saddle points, only two contribute to the leading behavior.

2. An integral representation of the second Airy function $\mathrm{Bi}(x)$ is given by

$$\mathrm{Bi}(x) = \frac{1}{2\pi}\int_{C_+} dt\, e^{xt - t^3/3} + \frac{1}{2\pi}\int_{C_-} dt\, e^{xt - t^3/3},$$

where $C_\pm$ is a contour which originates at $\infty e^{\pm 2\pi i/3}$ and terminates at $+\infty$. Using the method of steepest descents, find the leading asymptotic behavior as $x \to +\infty$.

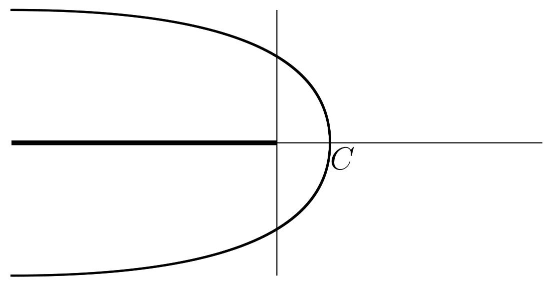

Figure 1: Contour C used for defining the Hankel representation of the gamma function.

3. One representation for the gamma function is
$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i}\int_C e^t t^{-z}\, dt$$
where the contour of integration is as shown in Figure 1. Use the method of steepest descents to derive Stirling's approximation,
$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}, \quad x \to +\infty.$$

4. The Airy function has the asymptotic expansion
$$\mathrm{Ai}(x) \sim \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-2x^{3/2}/3}[1 + O(x^{-3/2}) + O(x^{-3}) + \ldots].$$
Fill in the steps followed in class, and calculate the $O(x^{-3/2})$ and the $O(x^{-3})$ terms.

5. Using the integral representation for the Hankel function of the first kind
$$H_\nu^{(1)}(z) = \frac{e^{-i\nu\pi/2}}{\pi}\int_{-\pi/2+i\infty}^{\pi/2-i\infty} d\phi\, e^{i(z\cos\phi+\nu\phi)},$$
derive Debye's asymptotic expansion for $\tan\alpha > 0$ and ν large and positive:
$$H_\nu^{(1)}(\nu\sec\alpha) \sim \sqrt{\frac{2}{\pi\nu\tan\alpha}}e^{-i\pi/4}e^{i\nu(\tan\alpha-\alpha)}\left(1 + \frac{u_1(\cot\alpha)}{i\nu} + O(1/\nu^2)\right),$$
where
$$u_1(t) = \frac{3t+5t^3}{24}.$$