Physics 5013. Homework 3
Due Friday, September 16, 2011

September 9, 2011

1. Show that
$$1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}$$
is positive when $x=25/16$, and that
$$1-\frac{x^2}{2}+\frac{x^4}{24}$$
vanishes when $x=(6-2\sqrt{3})^{1/2}=1.59245\ldots$; and deduce that $3.125<\pi<3.185$.

2. Derive the expressions given in class for the inverse hyperbolic and trigonometric function. [Hint: Express the hyperbolic and trig functions in terms of exponentials, and solve for the exponent.]

3. Show that, with the cut lines given in class, the inverse hyperbolic and trig functions are single-valued.

4. Derive the following expansions of the Debye functions:
$$\int_0^x \frac{t^n\,dt}{e^t-1}=x^n\left[\frac{1}{n}-\frac{x}{2(n+1)}+\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{(2k+n)(2k)!}\right],\quad |x|<2\pi,\ n\geq 1,$$
$$\int_x^\infty \frac{t^n\,dt}{e^t-1} = \sum_{k=1}^{\infty} e^{-kx}\left[\frac{x^n}{k}+\frac{nx^{n-1}}{k^2}+\frac{n(n-1)x^{n-2}}{k^3}+\cdots+\frac{n!}{k^{n+1}}\right],$$
$$x>0,\ n\geq 1.$$
The complete integral from 0 to ∞ equals $n!\,\zeta(n+1)$.

5. Derive the following Bernoulli number series for the Euler-Macheroni constant

$$\gamma = \sum_{s=1}^{n} s^{-1} - \ln n - \frac{1}{2n} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)n^{2k}}.$$

Euler's constant γ is defined by

$$\gamma = \lim_{N\to\infty} \left[\sum_{k=1}^{N} \frac{1}{k} - \ln N \right].$$

[Hint: Apply the Euler-Maclaurin integration formula to $f(x) = x^{-1}$ over the range $[n, N]$.]

6. Use the Euler-Maclaurin sum formula to "evaluate" the sum

$$\sum_{l=0}^{\infty}{}' e^{-l^2 t},$$

where the prime means that the $l = 0$ term is only counted with half weight, to be

$$\frac{1}{2}\sqrt{\frac{\pi}{t}}.$$

[This result is, in fact, valid only for small t.]

7. The Euler numbers E_n are defined by the Taylor series expansion around $t = 0$ of $1/\cosh(t/2)$:

$$\frac{1}{\cosh(t/2)} = \sum_{n=0}^{\infty} E_n \frac{(t/2)^n}{n!}.$$

   (a) Which E_n's are zero?
   (b) Compute the first 3 nonzero E_n's.
   (c) For what t is $1/\cosh(t/2)$ singular?
   (d) Using the fact that the distance from the origin to the nearest singularity is the radius of convergence, determine the radius of convergence of this series.
   (e) What can you say about the asymptotic behavior of E_n for $n \gg 1$? (Be as precise as possible.)