Chapter 9

Asymptotic Expansions

We will illustrate the notions with a couple of carefully chosen examples. For more detail, you are referred to C. M. Bender and S. A. Orzag, *Advanced Mathematical Methods for Physicists and Engineers: Asymptotic Methods and Perturbation Theory* (Springer, 1999).

9.1 The Airy Function

The Airy function, which occurs, for example, in various radiation problems, is defined by the integral

$$\begin{aligned}\pi \mathrm{Ai}(\zeta) &= \int_0^\infty dt\, \cos\left(\zeta t + \frac{1}{3}t^3\right) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} dt\, e^{i(\zeta t + t^3/3)}. \end{aligned} \tag{9.1}$$

Let $z = it$; then this integral can also be given as

$$\mathrm{Ai}(\zeta) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\, e^{\zeta z - z^3/3}, \tag{9.2}$$

where the path of integration is along the imaginary axis. Now, to this point, this integral has only a formal existence, since the magnitude of the integrand is unity. However, if we distort the contour to C, as shown in Fig. 9.1, which passes through the origin, but is asymptotic to the lines $\arg z = \pm 2\pi/3$, we obtain a convergent integral since

$$z^3 = \left(\rho e^{\pm i2\pi/3}\right)^3 = \rho^3 > 0. \tag{9.3}$$

This deformation of the contour is permissible because the contributions of the arcs at infinity, connecting the ends of C to the imaginary axis, are negligible.

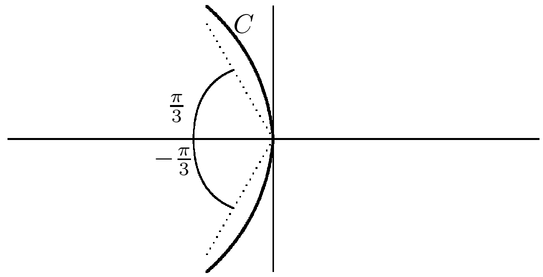

Figure 9.1: Contour C used to define the Airy function in Eq. 9.5.

That is, if $z = Re^{i\theta}$, $R \to \infty$, we have

$$\operatorname{Re} z^3 = R^3 \cos 3\theta > 0 \quad \text{if} \quad \frac{2\pi}{3} \geq \theta > \frac{\pi}{2} \quad \text{or if} \quad -\frac{2\pi}{3} \leq \theta < -\frac{\pi}{2}, \tag{9.4}$$

so the integrand is exponentially small there.[1] Thus the final definition of the Airy function is

$$\operatorname{Ai}(\zeta) = \frac{1}{2\pi i} \int_C dz\, e^{\zeta z - z^3/3}. \tag{9.5}$$

We now want to find a useful approximation to this integral valid for $|\zeta|$ large. To do so we note that the exponent, and its first two derivatives, are as functions of z,

$$\phi(z) = \zeta z - \frac{1}{3} z^3, \tag{9.6a}$$

$$\phi'(z) = \zeta - z^2, \tag{9.6b}$$

$$\phi''(z) = -2z, \tag{9.6c}$$

so that $\phi(z)$ has vanishing derivative when $z = \pm\sqrt{\zeta}$. (For definiteness, we shall suppose that ζ is real.) Since the integrand in the integral defining the Airy function is entire, we can deform C so that it passes through one of these points, say $z = -\sqrt{\zeta}$, as shown in Fig. 9.2. The reason we choose the contour C to pass through the stationary point $z = -\sqrt{\zeta}$ is that there the second derivative is positive, so that a curve whose tangent is parallel to the imaginary axis will pass through a maximum rather than a minimum. In particular, let us choose C so that ϕ is real everywhere along the path. Then for $\xi = z + \sqrt{\zeta}$ small, we can expand

$$\phi(z) \approx \phi(z = -\sqrt{\zeta}) + \phi''(z = -\sqrt{\zeta}) \frac{\xi^2}{2} = -\frac{2}{3} \zeta^{\frac{3}{2}} + 2\zeta^{\frac{1}{2}} \frac{\xi^2}{2}. \tag{9.7}$$

Requiring $\operatorname{Im} \phi = 0$ implies ξ be either real or imaginary. We choose the latter, as indicated in Fig. 9.2, so that ϕ will have a maximum at $z = -\sqrt{\zeta}$ on the path. This path is called *the path of steepest descents.*

[1] This argument fails in the immediate vicinity of the imaginary axis, reflecting the ill-defined nature of Eq. (9.2). A distortion so that $|\arg z| > \frac{\pi}{2}$ must be supplied in any case.

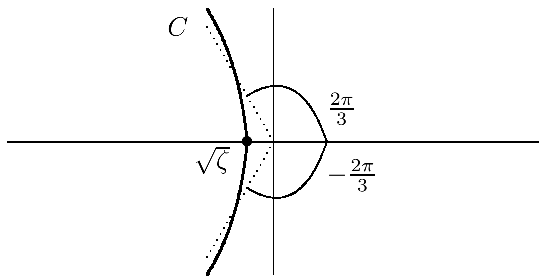

Figure 9.2: Deformed contour C which passes through the saddle point.

Note that in the perpendicular direction, along the real axis, the function is a minimum at the stationary point. Thus, the stationary point is a saddle point, and this method is also referred to as the *saddle point method.*

The reason for choosing C to be the path of steepest descents is that, for large $|\zeta|$, most of the contribution comes from the immediate neighborhood of the saddle point. Then we can make use of the approximation above, so that we approximate the Airy function by

$$\mathrm{Ai}(\zeta) \sim \frac{1}{2\pi i} e^{-\frac{2}{3}\zeta^{\frac{3}{2}}} \int_C d\xi\, e^{\sqrt{\zeta}\xi^2}, \tag{9.8}$$

where the integral is just a Gaussian one,

$$\int_C d\xi\, e^{\sqrt{\zeta}\xi^2} = \zeta^{-\frac{1}{4}} \int_{-i\infty}^{i\infty} du\, e^{u^2} = i\zeta^{-\frac{1}{4}} \int_{-\infty}^{\infty} dt\, e^{-t^2} = i\zeta^{-\frac{1}{4}}\sqrt{\pi}. \tag{9.9}$$

Thus we obtain the leading asymptotic behavior of the Airy function

$$\mathrm{Ai}(\zeta) \sim \frac{1}{2\sqrt{\pi}} \zeta^{-\frac{1}{4}} e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}, \quad \zeta \to \infty. \tag{9.10}$$

This result is actually valid for complex values of ζ subject to the restriction

$$|\arg \zeta| < \pi. \tag{9.11}$$

This asymptotic approximation is really quite good for modest ζ as Fig. 9.3 shows.

9.1.1 Asymptotic series

Let us calculate the corrections to this result. We return to Eq. (9.7) and keep the next term in ξ:

$$\phi(z) = -\frac{2}{3}\zeta^{3/2} + \zeta^{1/2}\xi^2 - \frac{1}{3}\xi^3, \tag{9.12}$$

which is exact in this case. Thus the Airy function is exactly represented by the integral

$$\mathrm{Ai}(\zeta) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\xi\, e^{-\frac{2}{3}\zeta^{3/2}} e^{\zeta^{1/2}\xi^2} e^{-\frac{1}{3}\xi^3}. \tag{9.13}$$

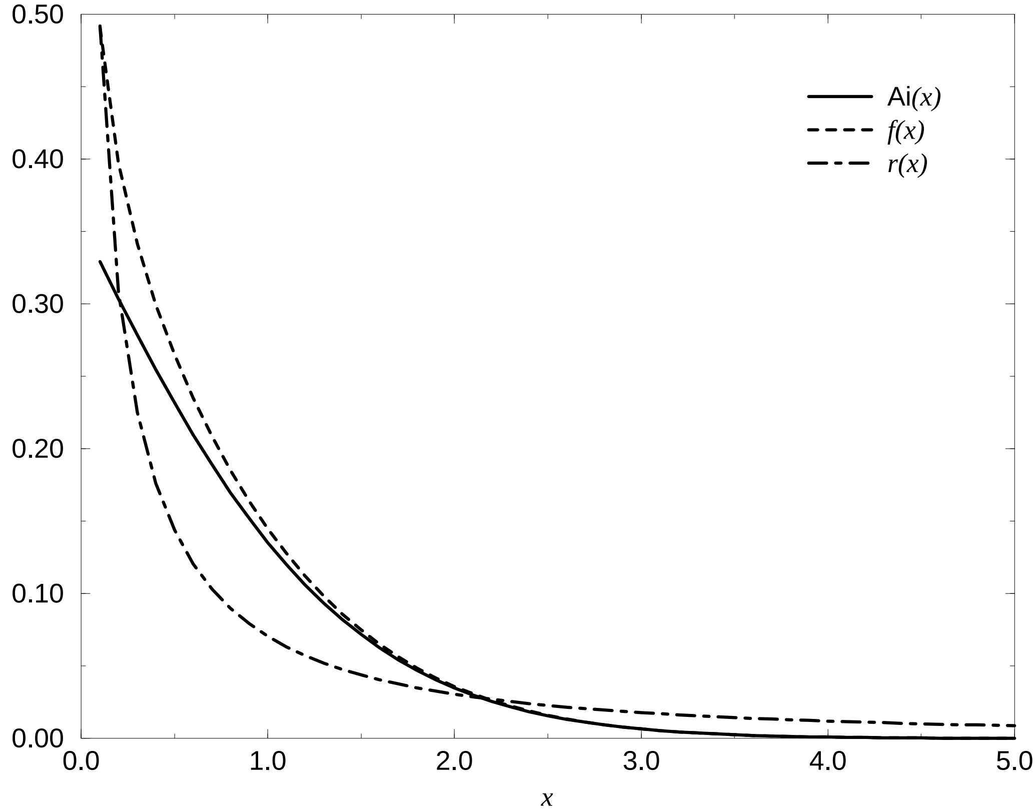

Figure 9.3: The Airy function $\mathrm{Ai}(x)$ compared with the asymptotic approximation (9.10), denoted $f(x)$, and the relative error of the latter, denoted $r(x)$. The error is less than 10% even for x as small as 1.

We approximate this by expanding the last exponential, since for large ζ the integrand is dominated by small ξ. Expanding out to fourth order, and omitting odd terms, we have after substituting $\xi = iu\zeta^{-1/4}$:

$$\mathrm{Ai}(\zeta) \sim \frac{\zeta^{-1/4}}{2\pi} e^{-\frac{2}{3}\zeta^{3/2}} \int_{-\infty}^{\infty} du\, e^{-u^2} \left(1 - \frac{1}{18}\frac{u^6}{\zeta^{3/2}} + \frac{1}{24}\frac{1}{81}\frac{u^{12}}{\zeta^3} + \dots\right). \tag{9.14}$$

The integrals may be evaluated starting from

$$\int_{-\infty}^{\infty} du\, e^{-\lambda u^2} = \sqrt{\frac{\pi}{\lambda}}, \tag{9.15}$$

so

$$\int_{-\infty}^{\infty} du\, u^{2k} e^{-\lambda u^2} = \left(-\frac{d}{d\lambda}\right)^k \int_{-\infty}^{\infty} du\, e^{-\lambda u^2} = \sqrt{\pi}\frac{(2k-1)!!}{2^k}\frac{1}{\lambda^{(2k+1)/2}}. \tag{9.16}$$

Thus, the two leading corrections to the asymptotic expression for the Airy function given in Eq. (9.10) are

$$\mathrm{Ai}(\zeta) \sim \frac{1}{2\sqrt{\pi}}\zeta^{-1/4} e^{-\frac{2}{3}\zeta^{3/2}} \left[1 - \frac{5}{48}\frac{1}{\zeta^{3/2}} + \frac{385}{4608}\frac{1}{\zeta^3} + \dots\right], \tag{9.17}$$

which is the beginning of an asymptotic series expansion in powers of $\zeta^{-3/2}$.

9.2 Synchrotron Radiation

A charged particle moving in a circular orbit emits electromagnetic radiation called (for the machine in which such radiation was first observed) *synchrotron radiation.* For details of the theory, see, for example, J. Schwinger, L. L. De-Raad, Jr., K. A. Milton, and W.-y. Tsai, *Classical Electrodynamics* (Perseus, 1998), p. 401 ff. In particular, the power radiated in the mth harmonic of the frequency of revolution of the charged particle moving in a circle with speed $v = \beta c$ is, in part, proportional to

$$J'_{2m}(2m\beta) = -\int_0^{\pi} \frac{d\phi}{\pi} \sin\phi\, \sin 2m(\beta\sin\phi - \phi). \tag{9.18}$$

In the ultrarelativistic limit when $\beta \to 1$, most of the radiation occurs for large harmonic numbers, $m \gg 1$, and the main contribution comes from the region near $\phi = 0$. Therefore, we may expand the integrand in Eq. (9.18) as follows:

$$\begin{aligned}\sin\phi\, \sin 2m(\beta\sin\phi - \phi) &\approx \phi \sin 2m\left(\beta\left[\phi - \frac{\phi^3}{3!}\right] - \phi\right)\\ &= \phi\sin\left(2m\left[-\phi(1-\beta) - \frac{1}{6}\beta\phi^3\right]\right)\\ &\approx -\phi\sin\left(m\left[(1-\beta^2)\phi + \frac{1}{3}\phi^3\right]\right)\end{aligned}$$

$$= -\sqrt{1-\beta^2}\, x \sin\left[m(1-\beta^2)^{3/2}\left(x+\frac{1}{3}x^3\right)\right], \tag{9.19}$$

where we have introduced the change of scale

$$\phi = \sqrt{1-\beta^2}\, x. \tag{9.20}$$

As a result, in this limit, Eq. (9.18) can be approximated by[2]

$$\begin{aligned} J'_{2m}(2m\beta) &\sim (1-\beta^2)\int_0^\infty \frac{dx}{\pi} x \sin\left(m(1-\beta^2)^{3/2}\left(x+\frac{1}{3}x^3\right)\right) \\ &= \frac{(1-\beta^2)}{\pi}\mathrm{Im}\int_0^\infty dx\, x\, e^{im(1-\beta^2)^{3/2}(x+x^3/3)}. \end{aligned} \tag{9.21}$$

For m fixed and β approaching unity in such a way that $m(1-\beta^2)^{3/2} \ll 1$, the significant contribution to Eq. (9.21) comes from the region where x is large, and Eq. (9.21) reduces to

$$\begin{aligned} J'_{2m}(2m\beta) &\sim (1-\beta^2)\int_0^\infty \frac{dx}{\pi} x \sin\left(\frac{m}{3}(1-\beta^2)^{3/2}x^3\right) \\ &= \int_0^\infty \frac{d\phi}{\pi}\phi\, \sin\left(\frac{m}{3}\phi^3\right), \end{aligned} \tag{9.22}$$

where all reference to the speed of the particle has disappeared. By changing variables, we may write this as

$$\begin{aligned} J'_{2m}(2m) &\sim -\mathrm{Im}\int_0^\infty \frac{d\phi}{\pi}\phi\, e^{-im\phi^3/3} \\ &= -\mathrm{Im}\left(\frac{3}{m}\right)^{2/3}\frac{e^{-i\pi/3}}{\pi}\int_0^\infty dt\left(\frac{1}{3}t^{-2/3}\right)t^{1/3}e^{-t} \\ &= -\mathrm{Im}\left(\frac{3}{m}\right)^{2/3}\frac{\Gamma(2/3)}{3\pi}e^{-i\pi/3} \\ &= \frac{3^{1/6}}{2\pi}\frac{\Gamma(2/3)}{m^{2/3}}, \quad \text{for} \quad m \gg 1. \end{aligned} \tag{9.23}$$

In the above evaluation, we have used Cauchy's theorem to perform a change of contour, as shown in Fig. 9.4, and have used the definition of the gamma function (8.69). Notice that Eq. (9.23) is valid for m either integer or half-integer.

However, for sufficiently large m, the parameter $m(1-\beta^2)^{3/2}$ becomes large, and the integrand in Eq. (9.21) undergoes rapid oscillations in x except near the stationary points, which satisfy

$$\frac{d}{dx}\left(x+\frac{1}{3}x^3\right) = 1 + x^2 = 0; \tag{9.24}$$

[2] Evidently, this integral is related to that defining the Airy function, Eq. (9.1).

ϕ plane:

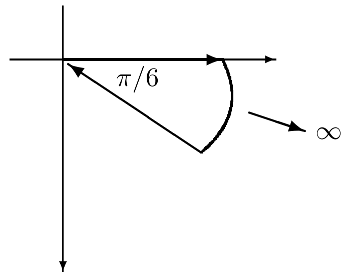

Figure 9.4: Change of contour used in evaluating Eq. (9.23).

Complex x plane

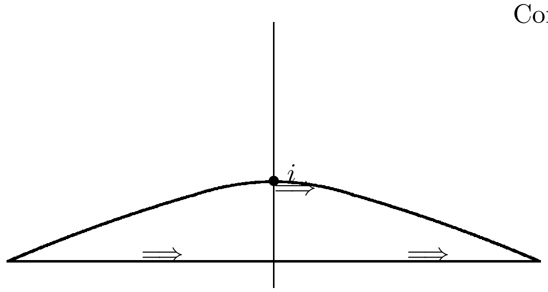

Figure 9.5: Stationary phase contour for evaluation of (9.21).

that is, the stationary phase points are located at

$$x = \pm i. \tag{9.25}$$

By extending the region of integration from $-\infty$ to $+\infty$, we evaluate Eq. (9.21) asymptotically by following the standard procedure of the saddle point method (or the method of steepest descents). We deform the contour of integration so that it passes through the stationary point $x = i$, because then the dominant contribution comes from the vicinity of that point. (See Fig. 9.5.) In the neighborhood of $x = i$, we let

$$x = i + \xi, \tag{9.26}$$

where ξ is real, to take advantage of the saddle point character. For arbitrary ξ

$$x + \frac{1}{3}x^3 = (i + \xi) + \frac{1}{3}(i + \xi)^3 = i\left(\frac{2}{3} + \xi^2\right) + \frac{1}{3}\xi^3, \tag{9.27}$$

so that for small ξ, if we drop the cubic term in ξ, the exponential factor in Eq. (9.21) becomes

$$e^{-\frac{2}{3}m(1-\beta^2)^{3/2}} e^{-m(1-\beta^2)^{3/2}\xi^2}, \tag{9.28}$$

which falls off exponentially on both sides of $x = i$. The resulting Gaussian integral in (9.21) leads to the following asymptotic form:

$$J'_{2m}(2m\beta) \sim \frac{1}{2}\frac{(1-\beta^2)^{1/4}}{\sqrt{\pi m}} e^{-\frac{2}{3}m(1-\beta^2)^{3/2}}, \quad m(1-\beta^2)^{3/2} \gg 1. \tag{9.29}$$

Thus, for very large harmonic numbers, the power spectrum[3] decreases exponentially in contrast to the behavior for smaller values of m where it increases like $m^{1/3}$. The transition between these two regimes occurs near the critical harmonic number, m_c , for which

$$m_c(1-\beta^2)^{3/2} \equiv 1, \tag{9.31}$$

or

$$m_c = (1-\beta^2)^{-3/2} = \left(\frac{E}{\mu c^2}\right)^3, \tag{9.32}$$

which uses the relativistic connection between the energy and the rest mass μ, $E = \mu c^2(1-\beta^2)^{-1/2}$. The bulk of the radiation is emitted with harmonic numbers near m_c. The qualitative shape of the spectrum is shown in Fig. 9.6.

9.2.1 First correction

Corrections to the formula (9.29) may be computed by retaining the ξ^3 term, but treating it as small, so the correction may be obtained by Taylor expanding the exponential:

$$\begin{aligned} J'_{2m}(2m\beta) &\sim \frac{1-\beta^2}{2\pi} \mathrm{Im}\, e^{-\frac{2}{3}m(1-\beta^2)^{3/2}} \int_{-\infty}^{\infty} d\xi\, e^{-m(1-\beta^2)^{3/2}\xi^2}(i+\xi) \\ &\quad \times \left(1 + im(1-\beta^2)^{3/2}\frac{\xi^3}{3} - \frac{1}{2}m^2(1-\beta^2)^3\frac{\xi^6}{9} + \dots\right) \\ &= \frac{1-\beta^2}{2\pi} e^{-\frac{2}{3}m(1-\beta^2)^{3/2}} (1-\beta^2)^{-3/4} m^{-1/2} \int_{-\infty}^{\infty} dt\, e^{-t^2} \\ &\quad \times \left(1 + \frac{1}{3}\frac{t^4}{m(1-\beta^2)^{3/2}} - \frac{1}{18}\frac{t^6}{m(1-\beta^2)^{3/2}} + \dots\right). \end{aligned} \tag{9.33}$$

Here we noted that the imaginary part only receives the contribution of the even terms in ξ, which are all that survive symmetric integration. Finally, the Gaussian integrals are evaluated according to

$$\int_{-\infty}^{\infty} dt\, t^{2n}\, e^{-t^2} = \int_0^{\infty} \frac{dx}{\sqrt{x}}\, x^n\, e^{-x} = \Gamma\left(n + \frac{1}{2}\right), \tag{9.34}$$

where

$$\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}, \quad \Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}. \tag{9.35}$$

[3]The power radiated into the mth harmonic by a particle of charge e moving in a circle of radius R with angular frequency ω_0 is given by

$$P_m = \frac{e^2}{R} m\omega_0 \left[2\beta^2 J'_{2m}(2m\beta) - (1-\beta^2)\int_0^{2m\beta} dx\, J_{2m}(x)\right] \tag{9.30}$$

The two terms in the square brackets have similar asymptotic behavior.

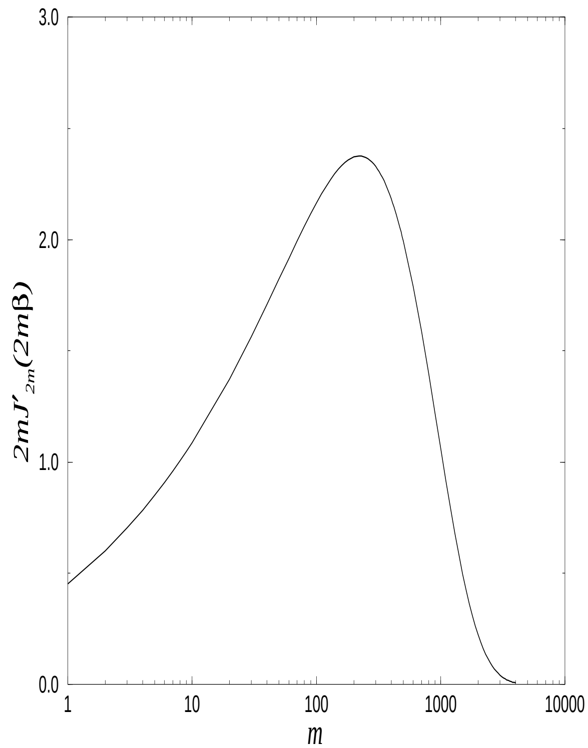

Figure 9.6: Sketch of power emitted into mth harmonic as a function of m. What is actually plotted is $2mJ'_{2m}(2m\beta)$ for $\beta = 0.99$. In this case $m_c = 356$.

Thus

$$J'_{2m}(2m\beta) = \frac{(1-\beta^2)^{1/4}}{2\sqrt{m\pi}} e^{-\frac{2}{3}m(1-\beta^2)^{3/2}} \times \left[1 + \frac{7}{48}\frac{1}{m(1-\beta^2)^{3/2}} + \mathcal{O}\left(\frac{1}{m^2(1-\beta^2)^3}\right)\right]. \quad (9.36)$$