Chapter 7

The Calculus of Residues

If $f(z)$ has a pole of order m at $z = z_0$, it can be written as Eq. (6.27), or

$$f(z) = \phi(z) = \frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \ldots + \frac{a_{-m}}{(z - z_0)^m}, \tag{7.1}$$

where $\phi(z)$ is analytic in the neighborhood of $z = z_0$. Now we have seen that if C encircles z_0 once in a positive sense,

$$\oint_C dz \frac{1}{(z - z_0)^n} = 2\pi i \delta_{n,1}, \tag{7.2}$$

where the Kronecker δ-symbol is defined by

$$\delta_{m,n} = \begin{cases} 0, \; m \neq n, \\ 1, \; m = n. \end{cases} . \tag{7.3}$$

Proof: By Cauchy's theorem we may take C to be a circle centered on z_0. On the circle, write $z = z_0 + re^{i\theta}$. Then the integral in Eq. (7.2) is

$$\frac{i}{r^{n-1}} \int_0^{2\pi} d\theta \, e^{i(1-n)\theta}, \tag{7.4}$$

which evidently integrates to zero if $n \neq 1$, but is $2\pi i$ if $n = 1$. QED.

Thus if we integrate the function (7.1) on a contour C which encloses z_0, while $\phi(z)$ is analytic on and within C, we find

$$\oint_C f(z) \, dz = 2\pi i a_{-1}. \tag{7.5}$$

Because the coefficient of the $(z - z_0)^{-1}$ power in the Laurent expansion of f plays a special role, we give it a name, the *residue* of $f(z)$ at the pole.

If C contains a number of poles of f, replace the contour C by contours α, β, γ, …encircling the poles singly, as shown in Fig. 7.1. The contour integral

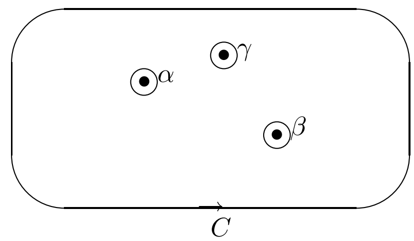

Figure 7.1: Integration of a function f around the contour C which contains only poles of f may be reduced to the integrals around subcontours α, β, γ, etc., each of which contains but a single pole of f.

around C may be distorted to a sum of disjoint ones around α, β, ..., so

$$\oint_C f(z)\,dz = \oint_\alpha f(z)\,dz + \oint_\beta f(z)\,dz + \dots, \tag{7.6}$$

and since each small contour integral gives $2\pi i$ times the reside of the single pole interior to that contour, we have established the *residue theorem*: If f be analytic on and within a contour C except for a number of poles within,

$$\oint_C f(z)\,dz = 2\pi i \sum_{\text{poles within } C} \text{residues}, \tag{7.7}$$

where the sum is carried out over all the poles contained within C.

This result is very usefully employed in evaluating definite integrals, as the following examples show.

7.1 Example 1

Consider the following integral over an angle:

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2}, \quad 0 < p < 1. \tag{7.8}$$

Let us introduce a complex variable according to

$$z = e^{i\theta}, \quad dz = ie^{i\theta}\,d\theta = iz\,d\theta, \tag{7.9}$$

so that

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right). \tag{7.10}$$

Therefore, we can rewrite the angular integral as an integral around a closed contour C which is a unit circle about the origin:

$$I = \oint_C \frac{dz}{iz} \frac{1}{1 - p\left(z + \frac{1}{z}\right) + p^2}$$

$$= \oint_C \frac{dz}{i} \frac{1}{z - p(z^2+1) + p^2 z}$$
$$= \frac{1}{i} \oint_C dz \frac{1}{(1-pz)(z-p)}. \tag{7.11}$$

The integrand exhibits two poles, one at $z = 1/p > 1$ and one at $z = p < 1$. Only the latter is inside the contour C, so since

$$\frac{1}{1-pz}\frac{1}{z-p} = \left(\frac{1}{z-p} + \frac{p}{1-pz}\right)\frac{1}{1-p^2}, \tag{7.12}$$

we have from the residue theorem

$$I = 2\pi i \frac{1}{i}\frac{1}{1-p^2} = \frac{2\pi}{1-p^2}. \tag{7.13}$$

Note that we could have obtained the residue without partial fractioning by evaluating the coefficient of $1/(z-p)$ at $z = p$:

$$\left.\frac{1}{1-pz}\right|_{z=p} = \frac{1}{1-p^2}. \tag{7.14}$$

This observation is generalized in the following.

7.2 A Formula for the Residue

If $f(z)$ has a pole of order m at $z = z_0$, the residue of that pole is

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0}. \tag{7.15}$$

The proof follows immediately from Eq. (7.1).

7.3 Example 2

This time we consider an integral along the real line,

$$I = \int_{-\infty}^{\infty} dx \frac{1}{(x^2+1)^3} = \lim_{R\to\infty}\int_{-R}^{R} dx \frac{1}{(x^2+1)^3}, \tag{7.16}$$

where we have made explicit the meaning of the upper and lower limits. We relate this to a contour integral as sketched in Fig. 7.2. Thus we have

$$\oint_C \frac{dz}{(z^2+1)^3} = \int_{-R}^{R} \frac{dx}{(x^2+1)^3} + \int_\Gamma \frac{dz}{(z^2+1)^3}, \tag{7.17}$$

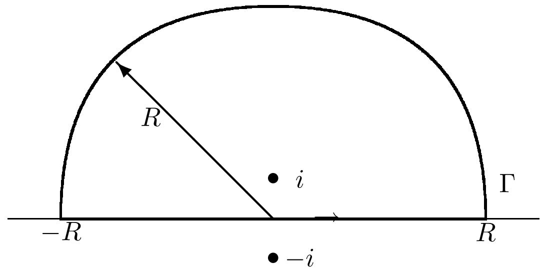

Figure 7.2: The closed contour C consists of the portion of the real axis between $-R$ and R, and the semicircle Γ of radius R in the upper half plane. Also shown in the figure are the location of the poles of the integrand in Eq. (7.17).

where we are to understand that the limit $R \to \infty$ is to be taken at the end of the calculation. It is easy to see that the integral over the large semicircle vanishes in this limit:

$$\int_\Gamma \frac{dz}{(z^2+1)^3} = \int_0^\pi \frac{R\,i\,e^{i\theta}d\theta}{\left(R^2e^{2i\theta}+1\right)^3} \to 0, \quad R \to \infty. \tag{7.18}$$

Hence the integral desired is just the closed contour integral,

$$I = \oint_C \frac{dz}{(z^2+1)^3} = 2\pi i(\text{residue at } i). \tag{7.19}$$

By the formula (7.15) the desired residue is

$$\begin{aligned} a_{-1} &= \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-i)^3\frac{1}{(z-i)^3(z+i)^3}\right]\Bigg|_{z=i} \\ &= \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{(z+i)^3}\Bigg|_{z=i} \\ &= \frac{1}{2!}\frac{(-3)(-4)}{(z+i)^5}\Bigg|_{z=i} \\ &= \frac{3}{16i}, \end{aligned} \tag{7.20}$$

so

$$I = \frac{3\pi}{8}. \tag{7.21}$$

7.4 Jordan's Lemma

The evaluation of a class of integrals depends upon this lemma. If $f(z) \to 0$ uniformly with respect to $\arg z$ as $|z| \to \infty$ for $0 \le \arg z \le \pi$, and $f(z)$ is analytic when $|z| > c > 0$ and $0 \le \arg z \le \pi$, then for $\alpha > 0$,

$$\lim_{\rho\to\infty}\int_{\Gamma_\rho} e^{i\alpha z} f(z)\,dz = 0, \tag{7.22}$$

where Γ_ρ is a semicircle of radius ρ above the real axis with center at the origin. (Cf. Fig. 7.2.)

Proof: Putting in polar coordinates,

$$\int_{\Gamma_\rho} e^{i\alpha z} f(z)\, dz = \int_0^\pi e^{i\alpha(\rho\cos\theta + i\rho\sin\theta)} f\left(\rho e^{i\theta}\right) \rho e^{i\theta} i\, d\theta. \tag{7.23}$$

If we take the absolute value of this equation, we obtain the inequality

$$\begin{aligned}\left|\int_{\Gamma_\rho} e^{i\alpha z} f(z)\, dz\right| &\le \int_0^\pi e^{-\alpha\rho\sin\theta} \left|f\left(\rho e^{i\theta}\right)\right| \rho\, d\theta \\ &< \varepsilon \int_0^\pi e^{-\alpha\rho\sin\theta} \rho\, d\theta,\end{aligned} \tag{7.24}$$

if $\left|f\left(\rho e^{i\theta}\right)\right| < \varepsilon$ for all θ when ρ is sufficiently large. (This is what we mean by going to zero uniformly for large ρ.) Now when

$$0 \le \theta \le \frac{\pi}{2}, \quad \sin\theta \ge \frac{2\theta}{\pi}, \tag{7.25}$$

which is easily verified geometrically. Therefore, the integral on the right-hand side of Eq. (7.24) is bounded as follows,

$$\begin{aligned}\int_0^\pi e^{-\alpha\rho\sin\theta} \rho\, d\theta &< 2\rho \int_0^{\pi/2} e^{-2\alpha\rho\theta/\pi}\, d\theta \\ &= \frac{\pi}{\alpha}\left(1 - e^{-\alpha\rho}\right).\end{aligned} \tag{7.26}$$

Hence

$$\left|\int_{\Gamma_\rho} e^{i\alpha z} f(z)\, dz\right| < \frac{\varepsilon\pi}{\alpha}\left(1 - e^{-\alpha\rho}\right) \tag{7.27}$$

may be made as small as we like by merely choosing ρ large enough (so $\varepsilon \to 0$). QED.

7.5 Example 3

Consider the integral

$$I = \int_0^\infty \frac{\cos x}{x^2 + a^2}\, dx. \tag{7.28}$$

The associated contour integral is

$$\oint_C \frac{e^{iz}}{z^2 + a^2}\, dz = \int_{-R}^R \frac{e^{ix}}{x^2 + a^2}\, dx + \int_\Gamma \frac{e^{iz}}{z^2 + a^2}\, dz, \tag{7.29}$$

where the contour Γ is a large semicircle of radius R centered on the origin in the upper half plane, as in Fig. 7.2. (The only difference here is that the pole inside the contour C is at ia.) The second integral on the right-hand side vanishes as $R \to \infty$ by Jordan's lemma. (Note carefully that this would not be true if we replace e^{iz} by $\cos z$ in the above.) Because only the even part of e^{ix} survives symmetric integration,

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \frac{1}{2}\oint_C \frac{e^{iz}}{z^2+a^2}\,dz \\ &= \frac{1}{2}2\pi i\frac{1}{2ia}e^{i(ia)} = \frac{\pi}{2a}e^{-a}. \end{aligned} \tag{7.30}$$

(Note that if C were closed in the lower half plane, the contribution from the infinite semicircle would not vanish. Why?)

7.6 Cauchy Principal Value

To this point we have assumed that the path of integration never encounters any singularities of the integrated function. On the contrary, however, let us now suppose that $f(x)$ has *simple poles* on the real axis, and try to attach meaning to

$$\int_{-\infty}^{\infty} f(x)\,dx. \tag{7.31}$$

For simplicity, suppose $f(z)$ has a simple pole at only one point on the real axis,

$$f(z) = \phi(z) + \frac{a_{-1}}{z - x_0}, \tag{7.32}$$

where $\phi(z)$ is analytic on the entire real axis. Then we define the (Cauchy) *principal value* of the integral as

$$P\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\delta\to 0+}\left(\int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{\infty} f(x)\,dx\right), \tag{7.33}$$

which means that the immediate neighborhood of the singularity is to be omitted symmetrically. The limit exists because $f(x) \approx a_{-1}/(x-x_0)$ near $x = x_0$, which is an odd function.

We can apply the residue theorem to such integrals by considering a deformed (indented) contour, as shown in Fig. 7.3. For simplicity, suppose the function falls off rapidly enough in the upper half plane so that

$$\int_\Gamma f(z)\,dz = 0, \tag{7.34}$$

where Γ is the "infinite" semicircle in the upper half plane. Then the integral around the closed contour shown in the figure is

$$\oint_C f(z)\,dz = P\int_{-\infty}^{\infty} f(x)\,dx - i\pi a_{-1}, \tag{7.35}$$

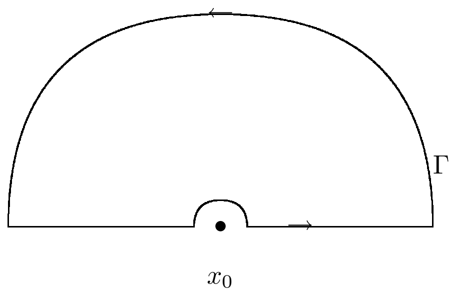

Figure 7.3: Contour which avoids the singularity along the real axis by passing above the pole.

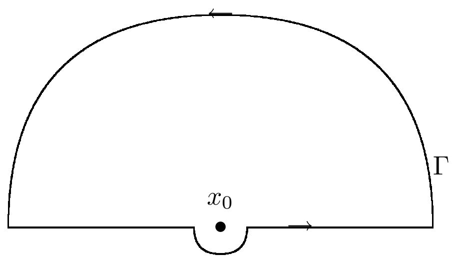

Figure 7.4: Contour which avoids the singularity along the real axis by passing below the pole.

where the second term comes from an explicit calculation in which the simple pole is half encircled in a negative sense (giving $-1/2$ the result if the pole were fully encircled in the positive sense). On the other hand, from the residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \sum_{\text{poles}\,\in\,\text{UHP}} (\text{residues}), \tag{7.36}$$

where UHP stands for upper half plane. Alternatively, we could consider a differently deformed contour, shown in Fig. 7.4. Now we have

$$\begin{aligned}\oint_C f(z)\,dz &= P\int_{-\infty}^{\infty} f(x)\,dx + i\pi a_{-1}\\ &= 2\pi i\left(\sum_{\text{poles}\,\in\,\text{UHP}} (\text{residues}) + a_{-1}\right),\end{aligned} \tag{7.37}$$

so in either case

$$P\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\text{poles}\,\in\,\text{UHP}} (\text{residues}) + \pi i a_{-1}, \tag{7.38}$$

where the sum is over the residues of the poles above the real axis, and a_{-1} is the residue of the simple pole on the real axis.

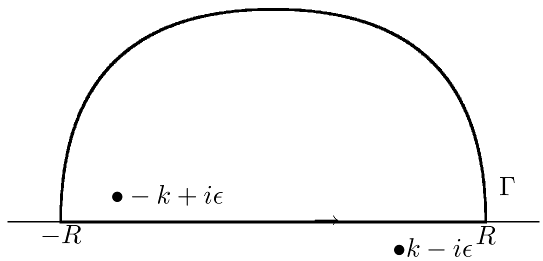

Figure 7.5: The closed contour C for the integral in Eq. (7.41).

Equivalently, instead of deforming the contour to avoid the singularity, one can displace the singularity, $x_0 \to x_0 \pm i\epsilon$. Then

$$\int_{-\infty}^{\infty} dx \frac{g(x)}{x - x_0 \mp i\epsilon} = P \int_{-\infty}^{\infty} dx \frac{g(x)}{x - x_0} \pm i\pi g(x_0), \tag{7.39}$$

if g is a regular function on the real axis. [Proof: Homework.]

7.7 Example 4

Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{iqx}}{q^2 - k^2 + i\epsilon}\, dq, \quad x > 0, \tag{7.40}$$

which is important in quantum mechanics. We can replace this integral by the contour integral

$$\oint_C \frac{e^{iqx}}{q^2 - k^2 + i\epsilon}\, dq, \quad x > 0, \tag{7.41}$$

where the closed contour C is shown in Fig. 7.5. The integral over the "infinite" semicircle Γ is zero according to Jordan's lemma. By redefining ϵ, but not changing its sign, we write the integral as ($k = +\sqrt{k^2}$)

$$\begin{aligned} I = \oint_C dq \left[\frac{1}{q - (k - i\epsilon)} \frac{1}{q + (k - i\epsilon)}\right] e^{iqx} &= 2\pi i \frac{e^{iqx}}{q - (k - i\epsilon)}\bigg|_{q = -(k - i\epsilon)} \\ &= -\frac{\pi i}{k} e^{-ikx}, \end{aligned} \tag{7.42}$$

in the end taking $\epsilon \to 0$.

7.8 Example 5

We will consider two ways of evaluating

$$I = \int_0^{\infty} \frac{dx}{1 + x^3}. \tag{7.43}$$

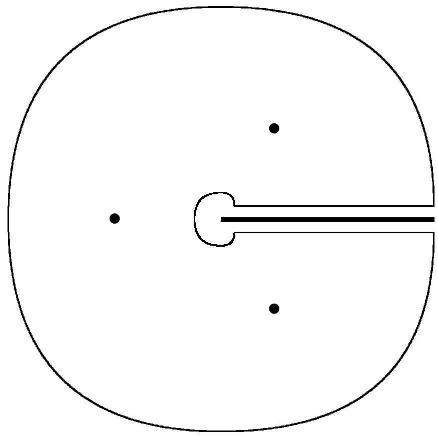

Figure 7.6: Contour C used in the evaluation of the integral (7.44). Shown also is the branch line of the logarithm along the $+z$ axis, and the poles of the integrand.

The integrand is not even, so we cannot extend the lower limit to $-\infty$. How can contour methods be applied?

7.8.1 Method 1

Consider the related integral

$$\oint_C \frac{\log z}{1+z^3}\,dz, \tag{7.44}$$

over the contour shown in Fig. 7.6. Here we have chosen the branch line of the logarithm to lie along the $+z$ axis; the discontinuity across it is

$$\text{disc}\,\log z = \log\rho - \log\rho e^{2i\pi} = -2i\pi. \tag{7.45}$$

The integral over the large circle is zero, as is the integral over the little circle:

$$\lim_{\rho\to\infty,0}\int_0^{2\pi}\frac{\log\rho e^{i\theta}}{1+\rho^2 e^{3i\theta}}\,\rho e^{i\theta}\,i\,d\theta = 0. \tag{7.46}$$

Therefore,

$$\begin{aligned} I &= -\frac{1}{2\pi i}\oint_C \frac{\log z}{1+z^3}\,dz \\ &= -\sum_{\text{poles inside }C}(\text{residues}). \end{aligned} \tag{7.47}$$

To find the sum of the residues, we note that the poles occur at the three cube roots of -1, namely, $e^{i\pi/3}$, $e^{i\pi}$, and $e^{5i\pi/3}$, so

$$\begin{aligned}
\sum(\text{residues}) &= \log e^{i\pi/3}\left(\frac{1}{e^{i\pi/3}-e^{i\pi}}\frac{1}{e^{i\pi/3}-e^{i5\pi/3}}\right) \\
&\quad + \log e^{i\pi}\left(\frac{1}{e^{i\pi}-e^{i\pi/3}}\frac{1}{e^{i\pi}-e^{i5\pi/3}}\right) \\
&\quad + \log e^{5i\pi/3}\left(\frac{1}{e^{5i\pi/3}-e^{i\pi/3}}\frac{1}{e^{5i\pi/3}-e^{i\pi}}\right) \\
&= i\frac{\pi}{3}\frac{1}{\left(\frac{1+\sqrt{3}i}{2}+1\right)}\frac{1}{\left(\frac{1+\sqrt{3}i}{2}-\frac{1-\sqrt{3}i}{2}\right)} \\
&\quad + i\pi\frac{1}{\left(-1-\frac{1+\sqrt{3}i}{2}\right)}\frac{1}{\left(-1-\frac{1-\sqrt{3}i}{2}\right)} \\
&\quad + \frac{i5\pi}{3}\frac{1}{\left(\frac{1-\sqrt{3}i}{2}-\frac{1+\sqrt{3}i}{2}\right)}\frac{1}{\left(\frac{1-\sqrt{3}i}{2}+1\right)} \\
&= \frac{i\pi}{3}\frac{1}{\sqrt{3}i}\frac{2}{3+\sqrt{3}i} + i\pi\frac{4}{9+3} + \frac{i5\pi}{3}\frac{-1}{\sqrt{3}i}\frac{2}{3-\sqrt{3}i} \\
&= \frac{\pi}{12}\left[\frac{2}{3\sqrt{3}}(3-\sqrt{3}i)+4i-\frac{10}{3\sqrt{3}}(3+\sqrt{3}i)\right] \\
&= -\frac{2\pi}{3\sqrt{3}},
\end{aligned} \tag{7.48}$$

or

$$I = \frac{2\pi}{3\sqrt{3}}. \tag{7.49}$$

7.8.2 Method 2

An alternative method which is simpler algebraically is the following. Consider

$$\oint_C \frac{dz}{z^3+1}, \tag{7.50}$$

where the contour C is shown in Fig. 7.7. The integral over the arc of the circle at "infinity," C_2, evidently vanishes as the radius of that circle goes to infinity. The integral over C_1 is the integral I. The integral over C_3 is

$$\int_{C_3}\frac{dz}{z^3+1} = \int_\infty^0 \frac{d(xe^{2i\pi/3})}{(xe^{2i\pi/3})^3+1} = -e^{2i\pi/3}I, \tag{7.51}$$

since $\left(e^{2i\pi/3}\right)^3 = 1$. Thus

$$\oint_C \frac{dz}{z^3+1} = I\left(1-e^{2\pi i/3}\right) = -Ie^{i\pi/3}2i\sin\frac{\pi}{3}. \tag{7.52}$$

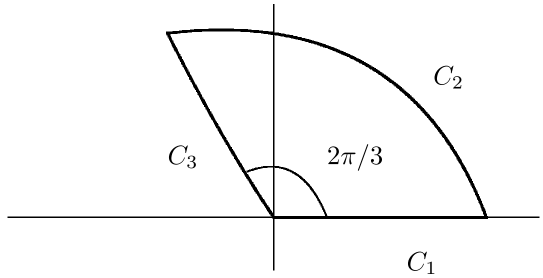

Figure 7.7: Contour used in the evaluation of Eq. (7.50).

The only pole of $1/(z^3+1)$ contained within C is at $z=e^{i\pi/3}$, the residue of which is

$$\frac{1}{e^{i\pi/3}-e^{i\pi}}\frac{1}{e^{i\pi/3}-e^{i5\pi/3}}=\frac{e^{-2\pi i/3}}{e^{-i\pi/3}-e^{i\pi/3}}\frac{e^{-3i\pi/3}}{e^{-2i\pi/3}-e^{2i\pi/3}}, \tag{7.53}$$

so

$$I=-\frac{2\pi i e^{-6\pi i/3}}{(2i)^3\left(\sin\frac{\pi}{3}\right)^2\sin\frac{2\pi}{3}}, \tag{7.54}$$

or since $\sin\frac{\pi}{3}=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$,

$$I=\frac{\pi}{4}\left(\frac{2}{\sqrt{3}}\right)^3=\frac{2\pi}{3\sqrt{3}}, \tag{7.55}$$

the same result (7.49) as found by method 1.

7.9 Example 6

Consider

$$I=\int_0^\infty \frac{x^{\mu-1}}{1+x}\,dx, \quad 0<\mu<1. \tag{7.56}$$

We may use the contour integral

$$\oint_C \frac{(-z)^{\mu-1}}{1+z}\,dz=\int_0^\infty e^{-i\pi(\mu-1)}\frac{x^{\mu-1}\,dx}{1+x}-\int_0^\infty e^{i\pi(\mu-1)}\frac{x^{\mu-1}\,dx}{1+x}, \tag{7.57}$$

where C is the same contour shown in Fig. 7.6, and because μ is between zero and one it is easily seen that the large circle at infinity and the small circle about the origin both give vanishing contributions. The pole now is at $z=-1$, so

$$\oint_C \frac{(-z)^{\mu-1}\,dz}{1+z}=2\pi i, \tag{7.58}$$

where the phase is measured from the negative real z axis. Thus

$$2\pi i=\left(e^{-i\pi(\mu-1)}-e^{i\pi(\mu-1)}\right)I=2iI\sin\pi\mu, \tag{7.59}$$

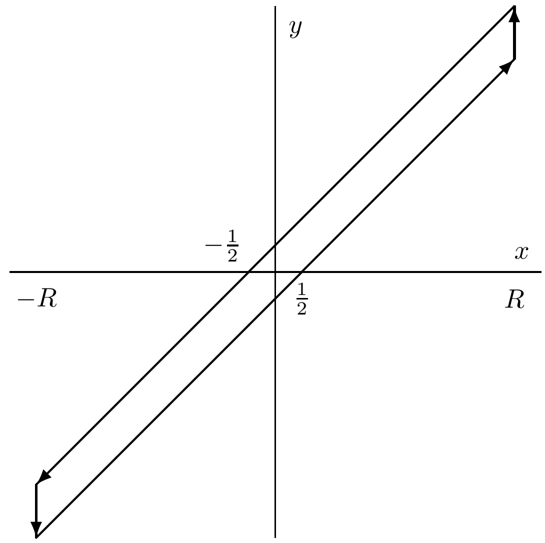

Figure 7.8: Contour C used in integral K, Eq. (7.62). Here the two lines making an angle of $\pi/4$ with respect to the real axis are closed with vertical lines at $x = \pm R$, where we will take the limit $R \to \infty$.

or

$$I = \frac{\pi}{\sin \pi\mu}. \tag{7.60}$$

7.10 Example 7

Here we demonstrate a method of evaluating the Gaussian integral,

$$J = \int_{-\infty}^{\infty} e^{-x^2}\, dx. \tag{7.61}$$

Consider the contour integral

$$K = \oint_C e^{i\pi z^2} \csc \pi z\, dz, \tag{7.62}$$

where C is the contour shown in Fig. 7.8. The equation for the two lines making angles of $\pi/4$ with respect to the real axis are

$$z = \pm\frac{1}{2} + \rho e^{i\pi/4}, \tag{7.63}$$

so

$$z^2 = \frac{1}{4} \pm \rho e^{i\pi/4} + i\rho^2. \tag{7.64}$$

Within the contour the only pole of $\csc \pi z$ is at $z = 0$, which has residue $1/\pi$, so by the residue theorem

$$K = 2\pi i \frac{1}{\pi} = 2i. \tag{7.65}$$

Directly, however,

$$\begin{aligned} K &= \int_{-\infty}^{\infty} e^{i\pi/4}\, d\rho\, \exp\left[i\pi\left(i\rho^2 + \rho e^{i\pi/4} + \frac{1}{4}\right)\right] \csc\pi\left(\rho e^{i\pi/4} + \frac{1}{2}\right) \\ &- \int_{-\infty}^{\infty} e^{i\pi/4}\, d\rho\, \exp\left[i\pi\left(i\rho^2 - \rho e^{i\pi/4} + \frac{1}{4}\right)\right] \csc\pi\left(\rho e^{i\pi/4} - \frac{1}{2}\right), \end{aligned} \tag{7.66}$$

since the vertical segments give exponentially vanishing contributions as $R \to \infty$. Combining these two integrals, we encounter

$$\begin{aligned} &\exp\left[i\pi\rho e^{i\pi/4}\right] \csc\pi\left(\rho e^{i\pi/4} + \frac{1}{2}\right) - \exp\left[-i\pi\rho e^{i\pi/4}\right] \csc\pi\left(\rho e^{i\pi/4} - \frac{1}{2}\right) \\ &\quad = 2\frac{\exp\left(i\pi\rho e^{i\pi/4}\right) + \exp\left(-i\pi\rho e^{i\pi/4}\right)}{\exp\left(i\pi\rho e^{i\pi/4}\right) + \exp\left(-i\pi\rho e^{i\pi/4}\right)} = 2, \end{aligned} \tag{7.67}$$

since $e^{\pm i\pi/2} = \pm i$. Hence

$$K = 2e^{i\pi/4}e^{i\pi/4}\int_{-\infty}^{\infty} d\rho\, e^{-\pi\rho^2} = \frac{2i}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2}, \tag{7.68}$$

so comparing with Eq. (7.65) we have for the Gaussian integral (7.61)

$$J = \sqrt{\pi}. \tag{7.69}$$

7.11 Example 8

Our final example is the integral

$$I = \int_0^{\infty} \frac{x\, dx}{1 - e^x}. \tag{7.70}$$

If we make the substitution $e^x = t$, this is the same as

$$I = \int_1^{\infty} \frac{\log t}{1 - t}\frac{dt}{t} = \int_1^{\infty} dt\, \log t\left[\frac{1}{t} + \frac{1}{1 - t}\right]. \tag{7.71}$$

If we make the further substitution in the first form of Eq. (7.71)

$$u = \frac{1}{t}, \quad \frac{du}{u} = \frac{dt}{t}, \tag{7.72}$$

we have

$$I = \int_0^1 \frac{\log \frac{1}{u}}{1 - \frac{1}{u}}\frac{du}{u} = \int_0^1 \frac{\log u}{1 - u}\, du, \tag{7.73}$$

If we average the two forms (7.71) and (7.73) we have

$$I = \frac{1}{2}\int_1^\infty dt\,\frac{\log t}{t} + \frac{1}{2}\int_0^\infty dt\,\frac{\log t}{1-t}. \tag{7.74}$$

The two integrals here separately are divergent, but the sum is finite. We regulate the two integrals by putting in a large t cutoff:

$$I = \frac{1}{2}\lim_{\Lambda\to\infty}\left[\int_1^\Lambda dt\,\frac{\log t}{t} + \int_0^\Lambda dt\,\frac{\log t}{1-t}\right]. \tag{7.75}$$

The first integral here is elementary,

$$\int_1^\Lambda dt\,\frac{\log t}{t} = \frac{1}{2}\log^2 t\bigg|_1^\Lambda = \frac{1}{2}\log^2\Lambda, \tag{7.76}$$

while the second is evaluated by considering

$$K = \oint_C dz\,\frac{\log^2 z}{1-z}, \tag{7.77}$$

where again C is the contour shown in Fig. 7.6. Now, however, the sole pole is on the positive real axis, so no singularities are contained within C, and hence by Cauchy's theorem $K = 0$.

This time the contribution of the large circle is not zero:

$$\begin{aligned}\int_0^{2\pi}\Lambda e^{i\theta}\,id\theta\,\frac{\log^2\Lambda e^{i\theta}}{1-\Lambda e^{i\theta}} &= -i\int_0^{2\pi} d\theta\,[\log\Lambda + i\theta]^2 \\ &= -i\left[2\pi\log^2\Lambda + 2i\frac{1}{2}(2\pi)^2\log\Lambda - \frac{1}{3}(2\pi)^3\right].\end{aligned} \tag{7.78}$$

The discontinuity of the $\log^2$ across the branch line is

$$\begin{aligned}\log^2 x - \log^2\left(xe^{2i\pi}\right) &= \log^2 x - (\log x + 2i\pi)^2 \\ &= -4i\pi\log x + 4\pi^2.\end{aligned} \tag{7.79}$$

Finally, notice that there is a contribution from the pole at $z = 1$ below the real axis (see Fig. 7.9): Explicitly, the contribution from the small semicircle below the pole is

$$\int_{2\pi}^{\pi} d\theta\,\frac{i\rho e^{i\theta}}{-\rho e^{i\theta}}\left[4i\pi\log\left(1+\rho e^{i\theta}\right) - 4\pi^2\right] = -4i\pi^3, \tag{7.80}$$

as $\rho\to 0$. The desired integral is obtained by taking the imaginary part,

$$\operatorname{Im} K = -4\pi\int_0^\Lambda dt\,\frac{\log t}{1-t} - \left[2\pi\log^2\Lambda - \frac{8\pi^3}{3}\right] - 4\pi^3 = 0, \tag{7.81}$$

so

$$\int_0^\Lambda dt\,\frac{\log t}{1-t} = -\frac{1}{2}\log^2\Lambda - \frac{\pi^2}{3}. \tag{7.82}$$

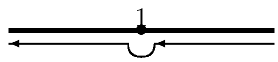

Figure 7.9: Portion of integral K, Eq. (7.77), corresponding to the integration below the cut on the real axis. The pole of the integrand at $z = 1$ contributes here because $\log(1 - i\epsilon) = 2i\pi$. Thus the contribution of the small semicircle to K is $+i\pi(2i\pi)^2 = -4i\pi^3$, in agreement with Eq. (7.80).

Thus averaging this with Eq. (7.76) we obtain

$$I = -\frac{\pi^2}{6}. \tag{7.83}$$

A slight check of this procedure comes from computing the real part of K:

$$\mathrm{Re}\, K = 4\pi^2 P \int_0^\Lambda \frac{dt}{1-t} + 4\pi^2 \log \Lambda = 0. \tag{7.84}$$