Chapter 5

Analytic Functions

5.1 The Derivative

Let $f(z)$ be a complex-valued function of the complex variable z. The *derivative* of f is defined as

$$f'(z) = \frac{df}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}, \tag{5.1}$$

if the limit exists and is independent of the way in which δz approaches zero. This is illustrated in Fig. 5.1

5.1.1 Examples

What is the derivative of z^n?

$$\begin{aligned} \frac{d}{dz} z^n &= \lim_{\delta z \to 0} \frac{(z+\delta z)^n - z^n}{\delta z} = \lim_{\delta z \to 0} \frac{n z^{n-1} \delta z}{\delta z} \\ &= n z^{n-1}. \end{aligned} \tag{5.2}$$

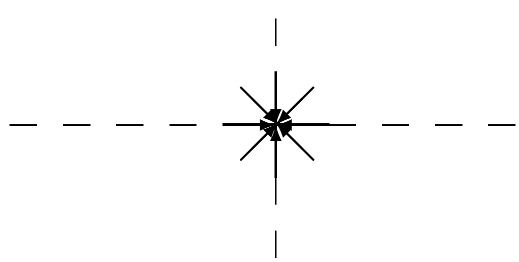

Figure 5.1: In the complex plane, δz, as indicated by the arrows in the figure, can approach zero from any direction.

Then, since e^z 1s represented by a power series which converges everywhere, and therefore converges uniformly in any finite bounded (compact) region, it is also differentiable everywhere,

$$\begin{aligned} \frac{d}{dz}e^z &= \frac{d}{dz}\sum_{n=0}^{\infty}\frac{1}{n!}z^n = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}z^{n-1} \\ &= e^z. \end{aligned} \tag{5.3}$$

The derivative of the exponential function is the function itself.

5.2 Analyticity

Whenever $f'(z_0)$ exists, f is said to be *analytic* (or *regular*, or *holomorphic*) at the point z_0. The function is analytic throughout a region in the complex plane if f' exists for every point in that region. Any point at which f' does *not* exist is called a *singularity* or *singular point* of the function f.

If $f(z)$ is analytic everywhere in the complex plane, it is called *entire*.

Examples

- $1/z$ is analytic except at $z = 0$, so the function is singular at that point.
- The functions z^n, n a nonnegative integer, and e^z are entire functions.

5.3 The Cauchy-Riemann Conditions

The Cauchy-Riemann conditions are necessary and sufficient conditions for a function to be analytic at a point.

Suppose $f(z)$ is analytic at z_0. Then $f'(z_0)$ may be obtained by taking δz to zero through purely real, or through purely imaginary values, for example.

If $\delta z = \delta x$, δx real, we have, upon writing f in terms of its real and imaginary parts, $f = u + iv$,

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z=z_0}. \tag{5.4}$$

On the other hand, if $\delta z = i\delta y$, δy real, we have similarly,

$$\begin{aligned} f'(z_0) &= \left(\frac{\partial u}{i\partial y} + i\frac{\partial v}{i\partial y}\right)_{z=z_0} \\ &= \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)_{z=z_0}. \end{aligned} \tag{5.5}$$

Since the derivative is independent of how the limit is taken, we can equate these two expression, meaning that they must have equal real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{5.6}$$

These are the *Cauchy-Riemann conditions.*

These conditions are not only necessary, but if the partial derivatives are continuous, they are sufficient to assure analyticity. Write

$$\begin{aligned} f(z+\delta z)-f(z) &= u(x+\delta x, y+\delta y)-u(x,y)+i[v(x+\delta x,y+\delta y)-v(x,y)] \\ &= u(x+\delta x,y+\delta y)-u(x,y+\delta y)+u(x,y+\delta y)-u(x,y) \\ &\quad +i[v(x+\delta x,y+\delta y)-v(x,y+\delta y)+v(x,y+\delta y)-v(x,y)] \\ &= \delta x\frac{\partial u}{\partial x}+\delta y\frac{\partial u}{\partial y}+i\left[\delta x\frac{\partial v}{\partial x}+\delta y\frac{\partial v}{\partial y}\right] \end{aligned} \tag{5.7}$$

which becomes, if the Cauchy-Riemann conditions hold

$$\begin{aligned} f(z+\delta z)-f(z) &= \delta x\frac{\partial u}{\partial x}-\delta y\frac{\partial v}{\partial x}+i\left[\delta x\frac{\partial v}{\partial x}+\delta y\frac{\partial u}{\partial x}\right] \\ &= (\delta x+i\delta y)\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right], \end{aligned} \tag{5.8}$$

so since $\delta z=\delta x+i\delta y$, we see

$$\frac{\delta f}{\delta z}\to\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \tag{5.9}$$

independently of how $\delta z\to 0$, so

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \tag{5.10}$$

exists.

Example

Consider the function z^* of z; that is, if $z=x+iy$, $z^*=x-iy$. The Cauchy-Riemann conditions never hold,

$$\frac{\partial x}{\partial x}=1\neq\frac{\partial(-y)}{\partial y}=-1, \tag{5.11}$$

so z^* is nowhere an analytic function of z.

5.4 Contour Integrals

Suppose we have a smooth path in the complex plane, extending from the point a to the point b. Suppose we choose points z_1, z_2,..., z_n-1 lying on the curve, and connect them by straight-line segments. Likewise connect $a=z_0$ with z_1 amd $b=z_n$ with z_{n-1}. See Fig. 5.2. Then the *contour integral* of a function f is defined by the following limit,

$$\int_{\substack{a\\C}}^b f(z)\,dz=\lim_{\substack{\Delta z_i\to 0\\ n\to\infty}}\sum_{i=1}^n f(z_i)\Delta z_i,\quad \Delta z_i=z_i-z_{i-1}, \tag{5.12}$$

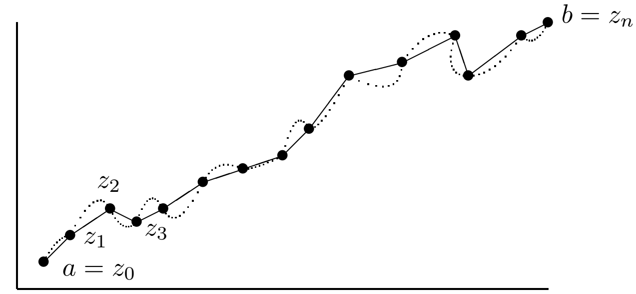

Figure 5.2: Path C in the complex plane approximated by a series of straight-line segments.

and the limit taken is one in which the number n of straight-line segments goes to infinity, while the length of the largest one goes to zero. Whenever this limit exists, independently of how it is taken, the integral exists. Note that in general the integral depends on the path C, as well as on the endpoints.

Example

Consider

$$\oint_K \frac{dz}{z} \tag{5.13}$$

where K is a circle about the origin, of radius r. (The circle on the integral sign signifies that the path of integration is closed.) From the polar representation of complex numbers, we may write

$$z = re^{i\theta}, \tag{5.14a}$$

so since r is fixed on K, we have

$$dz = re^{i\theta} i \, d\theta. \tag{5.14b}$$

Let us assume that the integration is carried out in a positive (counterclockwise) sense, so then

$$\oint_K \frac{dz}{z} = i \int_0^{2\pi} d\theta = 2\pi i, \tag{5.15}$$

which is independent of the value of r.

5.5 Cauchy's Theorem

Chauchy's theorem states that if $f(z)$ is analytic at all points on and inside a closed contour C, then the integral of the function around that contour vanishes,

$$\oint_C f(z) \, dz = 0. \tag{5.16}$$

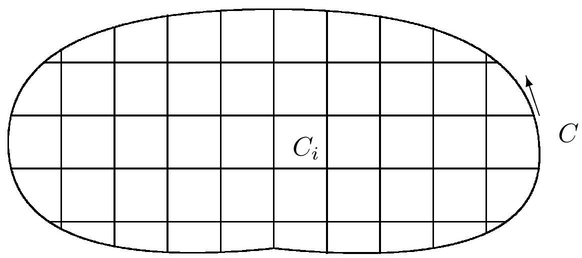

Figure 5.3: The integral around the contour C may be replaced by the sum of integrals around the subcontours C_i.

Proof: Subdivide the region inside the contour in the manner shown in Fig. 5.3. Obviously

$$\oint_C f(z)\,dz = \sum_i \oint_{C_i} f(z)\,dz, \tag{5.17}$$

where C_i is the closed path around one of the mesh elements, since the contribution from the side common to two adjacent subcontours evidently cancels, leaving only the contribution from the exterior boundary. Now because f is analytic throughout the region, we may write for small δz

$$f(z + \delta z) = f(z) + \delta z\, f'(z) + \mathcal{O}(\delta z^2), \tag{5.18}$$

where $\mathcal{O}(\delta z^2)$ means only that the remainder goes to zero faster that δz. We apply this result by assuming that we have a fine mesh subdividing C—we are interested in the limit in which the largest mesh element goes to zero. Let z_i be a representative point within the ith mesh element (for example, the center). Then

$$\oint_{C_i} f(z)\,dz = f(z_i)\oint_{C_i} dz + f'(z_i)\oint_{C_i}(z - z_i)\,dz + \oint_{C_i}\mathcal{O}((z - z_i)^2)\,dz. \tag{5.19}$$

Now it is easily seen that for an arbitrary contour C_i

$$\oint_{C_i} dz = \oint_{C_i}(z - z_i)\,dz = 0, \tag{5.20}$$

so if the length of the cell is ε,

$$\oint_{C_i} f(z)\,dz = \mathcal{O}(\varepsilon^3) = A_i\mathcal{O}(\varepsilon), \tag{5.21}$$

which is to say that the integral around the ith cell goes to zero faster than the area A_i of the ith cell. Thus the integral required is

$$\oint_C f(z)\,dz = \sum_i A_i\mathcal{O}(\varepsilon) = A\mathcal{O}(\varepsilon), \tag{5.22}$$

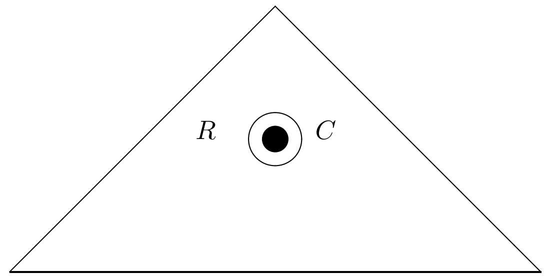

Figure 5.4: A multiply connected region R consisting of the area within a triangle but outside of an circular region. The closed contour C cannot be continuously deformed to a point without crossing into the disk, which is outside the region R.

where A is the finite area contained within the contour C. As the subdivision becomes finer and finer, $\varepsilon \to 0$ and so

$$\oint_C f(z)\,dz = 0. \tag{5.23}$$

To state a more general form of Cauchy's theorem, we need the concept of a simply connected region. A *simply connected region* R is one in which any closed contour C lying in R may be continuously shrunk to a point without ever leaving R. Fig. 5.4 is an illustration of a multiply connected region. C lies entirely within R, yet it cannot be shrunk to a point because of the excluded region inside it.

We can now restate Cauchy's theorem as follows: *If f is analytic in a simply connected region R then*

$$\oint_C f(z)\,dz = 0 \tag{5.24}$$

for any closed contour C in R.

That simple connectivity is required here is seen by the example of the function $1/z$, which is analytic in any region excluding the origin.

Here is another proof of Cauchy's theorem, as given in the book by Morse and Feshbach. If the closed contour C lies in a simply-connected region where $f'(z)$ exists then

$$\oint_C f(z)\,dz = 0. \tag{5.25}$$

Proof: Let us choose the origin to lie in the region of analyticity (if it does not, change variables so that $z = 0$ lies within C). Define

$$F(\lambda) = \lambda \oint_C f(\lambda z)\,dz. \tag{5.26}$$

Then the derivative of this function of λ is

$$F'(\lambda) = \oint_C f(\lambda z)\,dz + \lambda \oint_C z f'(\lambda z)\,dz$$

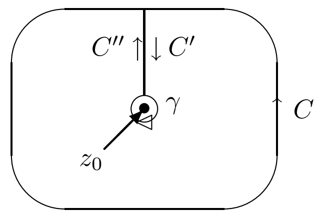

Figure 5.5: Distortion of a contour C to a small one γ encircling the singularity at z_0.

$$= \oint_C f(\lambda z)\,dz + z f(\lambda z)\Big|_{z=\text{beginning of }C}^{z=\text{end of }C} - \oint_C f(\lambda z)\,dz = 0, \quad (5.27)$$

where we have integrated by parts, because the function f is single valued. Thus $F(\lambda)$ is constant. But

$$F(0) = \lim_{\lambda\to 0} \lambda \oint_C f(\lambda z)\,dz = \lim_{\lambda\to 0} \oint_{\lambda C} f(z)\,dz = 0 \quad (5.28)$$

because $f(0)$ is bounded because f is analytic at the origin. (We have deformed the contour to an infinitesimal one about the origin.) Thus we conclude that $F(1) = 0$. This proves the theorem.

5.6 Cauchy's Integral Formula

If $f(z)$ is analytic on and within the closed contour C, and z_0 lies within C, then the value of f at z_0 is given in terms of its boundary values by

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz, \quad (5.29)$$

where the contour is traversed in the positive (counterclockwise) sense.

Proof: $f(z)/(z - z_0)$ is not analytic within C, so choose a contour inside of which this function is analytic, as shown in Fig. 5.5. Here we have connected the contour C to the small contour γ by two overlapping lines C', C'' which are traversed in opposite senses. Now $f(z)/(z - z_0)$ is analytic on the inside of the contour $C + C' + C'' + \gamma$. (By inside, we mean that if you follow the path in the direction indicated by the arrows, the inside is only your left, and the outside is on your right.) Thus, by Cauchy's theorem

$$\oint_{C+C'+C''+\gamma} \frac{f(z)}{z - z_0} dz = 0. \quad (5.30)$$

Now because we choose the lines C', C'' as overlapping, since f is continuous in the neighborhood of those lines those two integrals cancel,

$$\int_{C'+C''} \frac{f(z)}{z-z_0} dz = 0. \tag{5.31}$$

And since the circle γ may be chosen arbitrarily small

$$\oint_\gamma \frac{f(z)}{z-z_0} dz = f(z_0) \oint_\gamma \frac{dz}{z-z_0} = -2\pi i f(z_0), \tag{5.32}$$

since γ is traversed in a negative or clockwise sense. Thus the theorem (5.29) is proved.

(Implicit in the above is the assumption that the contour does not cross itself to wind around z_0 more than once. If this happens, Cauchy's formula is modified. See homework.)

It is now easily shown from the definition of the derivative that if f is analytic on and within C, we may express the derivative by

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz, \tag{5.33}$$

and in fact the nth derivative is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz. \tag{5.34}$$

That is, if f is analytic, so is its derivative. An analytic function is infinitely differentiable, a property which is not true for a differentiable function of a real variable.

5.7 Morera's Theorem

The converse to Cauchy's theorem is the following:

If $f(z)$ is *continuous* in a region R, and for all contours C lying in R

$$\oint_C f(z)\, dz = 0, \tag{5.35}$$

then $f(z)$ is analytic throughout R.

Proof: If f satisfies the above hypotheses, then the integral

$$\int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1) \tag{5.36}$$

is a function of the endpoints only, and not of the path, as is evident from Fig. 5.6. But now the function F has a unique derivative,

$$F'(z) = f(z), \tag{5.37}$$

so that $F(z)$ is analytic. Hence, so is its derivative $f(z)$. QED.

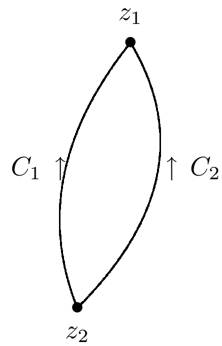

Figure 5.6: Two paths C_1 and C_2 connecting the point z_1 with the point z_2. Because $\oint_{C_1-C_2} f(z)\,dz = 0$, we conclude that $\int_{z_1 C_1}^{z_2} f(z)\,dz = \int_{z_1 C_2}^{z_2} f(z)\,dz$.

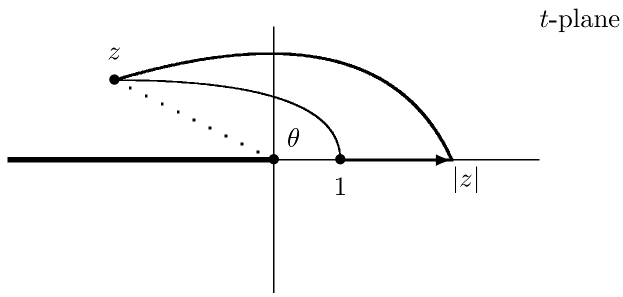

Figure 5.7: Path of integration in the cut t plane used in defining the logarithm in Eq. (5.38).

5.8 The Logarithm

An alternative definition to that given in Sec. 3.2 is given by the path integral

$$\log z = \int_1^z \frac{dt}{t}, \tag{5.38}$$

over any contour connecting 1 with z which does not cross the cut line shown in Fig. 5.7. The cut is present so the contour cannot encircle the singularity of the integrand at $t = 0$. Because the arg function must be single-valued, the cut supplies the restriction

$$-\pi < \arg(z) \leq \pi. \tag{5.39}$$

The last equality means for negative z we approach the cut from above.

Since the integral is path independent, we may chose the path to consist of a segment along the positive z axis and an arc of a circle, as also shown in Fig. 5.7. Then the logarithm may be written as

$$\log z = \int_1^{|z|} \frac{dt}{t} + \int_0^{\theta} \frac{|z|\, i\, e^{i\theta'}\, d\theta'}{|z|\, e^{i\theta'}}$$

$$\begin{aligned} &= \log|z| + i\theta \\ &= \log|z| + i \arg z, \end{aligned} \tag{5.40}$$

which coincides with the previous definition.

The logarithm is analytic in the cut plane, and its derivative is

$$\frac{d}{dz}\log z = \frac{1}{z}. \tag{5.41}$$

If $\xi = \log z$, define the inverse function by $z = \exp\xi$. Since when $z = 1$, $\xi = 0$, we have

$$\exp(0) = 1. \tag{5.42}$$

Also we have

$$\frac{d}{d\xi}\exp\xi = \frac{dz}{d\xi} = \frac{dz}{d\log z} = z = \exp\xi. \tag{5.43}$$

These two properties uniquely define the exponential function.

5.9 A Theorem for Functions Represented by Series

Let us suppose that the function Φ defined by the series

$$\Phi(z) = \sum_{n=0}^{\infty} f_n(z) \tag{5.44}$$

converges *uniformly* on a closed contour C, and that each f_n is analytic on and within C. Then, on and within C

$$\Phi(z) = \sum_{n=0}^{\infty} f_n(z) \tag{5.45}$$

converges and Φ is analytic.

Proof: Since a uniformly convergent series may be integrated term by term, we have for z_0 within C

$$\begin{aligned} \frac{1}{2\pi i}\oint_C \frac{\Phi(z)}{z - z_0}dz &= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_C \frac{f_n(z)}{z - z_0}dz \\ &= \sum_{n=0}^{\infty} f_n(z_0), \end{aligned} \tag{5.46}$$

by Cauchy's integral formula. So this last sum exists; call it

$$\Phi(z_0) = \sum_{n=0}^{\infty} f_n(z_0). \tag{5.47}$$

Now $\Phi'(z_0)$ exists as well:

$$\Phi'(z_0) = \frac{1}{2\pi i} \oint_C \frac{\Phi(z)}{(z - z_0)^2} dz = \sum_{n=0}^{\infty} f_n'(z_0), \tag{5.48}$$

so Φ is analytic within C.