Add these two series:

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\ldots=\frac{3}{2}\ln 2. \tag{2.20}$$

Since the reciprocal of each integer occurs exactly once in the last series, we would be tempted to rearrange the series to obtain

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=\ln 2, \tag{2.21}$$

which is identical to the original series. There is an obvious contradiction here! In order to obtain the rearrangement (2.21), we have to go further and further out in the series (2.20), which apparently is not permissible.

2.4.2 A Theorem About Absolutely Convergent Series

Not only can absolutely convergent series be rearranged without changing their value, but they can be multiplied together term by term: If two series

$$S=\sum_{i=1}^{\infty}u_i, \tag{2.22a}$$

$$T=\sum_{i=1}^{\infty}v_i \tag{2.22b}$$

are both *absolutely* convergent, the series

$$P=\sum_{\substack{i=1\\j=1}}^{\infty}u_iv_j \tag{2.23}$$

formed from the product of their terms written in any order, is absolutely convergent, and has a value equal to the product of of the individual series,

$$P=ST. \tag{2.24}$$

2.5 Convergence Tests

The following tests can determine whether a given series is absolutely convergent or not.

2.5.1 Comparison test

If $b_n>0$ for all n and $\sum_{n=1}^{\infty}b_n$ is convergent, and if $|a_n|\le b_n$ for all n, then

$$\sum_{n=1}^{\infty}a_n \quad \text{is absolutely convergent.} \tag{2.25a}$$

Also, if $|a_n| \geq b_n > 0$ for all n, and $\sum_{n=1}^{\infty} b_n$ diverges, then

$$\sum_{n=1}^{\infty} a_n \quad \text{is } \textit{not} \text{ absolutely convergent.} \tag{2.25b}$$

2.5.2 Root test

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if from a certain term onward

$$\sqrt[n]{|a_n|} \leq q < 1, \tag{2.26}$$

where $q \geq 0$ is independent of n.

Proof: If the inequality holds, $|a_n| \leq q^n$. But $\sum_{n=1}^{\infty} q^n$ converges for $q < 1$, it being the geometric series, so by 2.5.1, $\sum_{n=1}^{\infty} |a_n|$ converges.

2.5.3 Ratio test

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if from a certain term onward

$$\left|\frac{a_{n+1}}{a_n}\right| \leq q < 1, \tag{2.27}$$

where $q \geq 0$ is independent of n.

Proof: Without loss of generality, we may assume the inequality holds for all n; otherwise, we renumber the $\{a_n\}$ sequence so that 1 labels the first term for which the inequality (2.27) holds. Then

$$\left|\frac{a_n}{a_1}\right| = \left|\frac{a_n}{a_{n-1}}\right| \left|\frac{a_{n-1}}{a_{n-2}}\right| \left|\frac{a_{n-2}}{a_{n-3}}\right| \cdots \left|\frac{a_2}{a_1}\right| \leq q^{n-1}. \tag{2.28}$$

Convergence is again assured by comparison with the geometric series. (Whether these tests are satisfied by the first few terms of a series is immaterial, since a finite number of terms of an infinite seris has no effect on the convergence.)

Example

When does $\sum_{n=1}^{\infty} nq^n$ converge? If we use the root test, we examine

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = |q| \lim_{n\to\infty} \sqrt[n]{n} = |q|,^{1} \tag{2.29a}$$

while if we use the ratio test, we look at

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |q| \lim_{n\to\infty} \frac{n+1}{n} = |q|. \tag{2.29b}$$

In either case, we see that the series is absolutely convergent if $|q| < 1$, and divergent otherwise.

---

[1] Because $\ln \sqrt[n]{n} = \frac{1}{n} \ln n$, which tends to zero as $n \to \infty$, $\sqrt[n]{n} \to 1$.

The following are refinements of the ratio test, which fails (that is, fails to reveal whether the tested series converges or not) when

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1. \tag{2.30}$$

For example, this indeterminate limit results for the case $a_n = 1/n$, which yields a divergent series, but also for $a_n = 1/(n\ln^2 n)$, which corresponds to a convergent sum (see Sec. 2.5.8).

2.5.4 Kummer's test

Choose a sequence of positive constants b_n. If

$$b_n\left|\frac{a_n}{a_{n+1}}\right| - b_{n+1} \geq C > 0, \tag{2.31}$$

for all $n \geq N$, where N and C are fixed numbers, then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges absolutely.} \tag{2.32}$$

On the other hand, if

$$b_n\left|\frac{a_n}{a_{n+1}}\right| - b_{n+1} \leq 0, \tag{2.33}$$

and

$$\sum_{n=1}^{\infty} b_n^{-1} \quad \text{diverges,} \tag{2.34}$$

then

$$\sum_{n=1}^{\infty} |a_n| \quad \text{diverges.} \tag{2.35}$$

Proof: If the inequality (2.31) holds, take $l \geq N$, so that

$$C|a_{l+1}| \geq b_l|a_l| - b_{l+1}|a_{l+1}|. \tag{2.36}$$

So we have the inequality

$$\sum_{l=N+1}^{n} |a_l| \leq \frac{b_N|a_N|}{C} - \frac{b_n|a_n|}{C} \leq \frac{b_N|a_N|}{C}. \tag{2.37}$$

Hence, the nth partial sum, for $n > N$, is

$$s_n = \sum_{i=1}^{n} |a_i| \leq \sum_{i=1}^{N} |a_i| + \frac{b_N|a_N|}{C}. \tag{2.38}$$

The right-hand side of this inequality is a constant, independent of n. Therefore, the positive sequence of increasing terms $\{s_n\}$ is bounded above, and consequently possesses a limit. The series is absolutely convergent.

If the inequality (2.33) holds,

$$|a_n| \geq \frac{|a_N| b_N}{b_n}, \quad n > N, \tag{2.39}$$

so since $\sum_{n=1}^\infty b_n^{-1}$ diverges, so does $\sum_{n=1}^\infty |a_n|$.

2.5.5 Raabe's test

Raabe's criterion for absolute convergence is

$$n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) \geq K > 1, \tag{2.40}$$

for all $n \geq N$, where N and K are fixed. And if

$$n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) \leq 1, \tag{2.41}$$

then

$$\sum_{n=1}^\infty |a_n| \quad \text{diverges.} \tag{2.42}$$

Proof: In Kummer's test put $b_n = n$.

2.5.6 Gauss' test

If

$$\left|\frac{a_n}{a_{n+1}}\right| = 1 + \frac{h}{n} + \frac{B(n)}{n^2}, \tag{2.43}$$

where h is a constant and the function $B(n)$ is bounded as $n \to \infty$, then $\sum_{n=1}^\infty |a_n|$ converges for $h > 1$ and diverges for $h \leq 1$.

Proof: For $h \neq 1$ we can use Raabe's test:

$$\lim_{n\to\infty} n\left(\frac{h}{n} + \frac{B(n)}{n^2}\right) = h. \tag{2.44}$$

For $h = 1$, Raabe's test is indeterminate. In that case use Kummer's test with $b_n = n \ln n$: for large n,

$$\begin{aligned}
&n \ln n \left(1 + \frac{h}{n} + \frac{B(n)}{n^2}\right) - (n+1)\ln(n+1) \\
&\approx n \ln n \left(1 + \frac{h}{n} + \frac{B(n)}{n^2}\right) - (n+1)\left(\ln n + \frac{1}{n}\right) \\
&\approx \left(h + \frac{B(n)}{n^2}\right) \ln n - \ln n - 1 \\
&\approx (h-1)\ln n - 1 < 0, \quad \text{if} \quad h \leq 1.
\end{aligned} \tag{2.45}$$

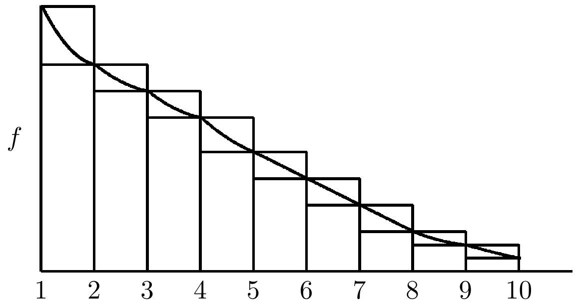

Figure 2.1: Bounds on a monotone series provided by an integral.

Because

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \quad \text{diverges} \tag{2.46}$$

(see homework), the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

2.5.7 Integral test

If $f(x)$ is a continuous, monotonically decreasing real function of x such that

$$f(n) = |a_n|, \tag{2.47}$$

then

$$\sum_{n=1}^{\infty} |a_n| \quad \text{converges if} \quad \int_1^{\infty} dx\, f(x) < \infty, \tag{2.48}$$

and diverges otherwise.

Proof: It is geometrically obvious that

$$\int_1^{\infty} dx\, f(x) < \sum_{n=1}^{\infty} f(n) < \int_1^{\infty} dx\, f(x) + f(1), \tag{2.49}$$

for this follows merely from the geometrical meaning of the integral as the area under the curve of the function. See Fig. 2.1.

2.5.8 Examples

- The Riemann zeta function is defined by the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} = \zeta(\alpha). \tag{2.50}$$

We can test for convergence using Gauss' test, by examining

$$\left(\frac{n+1}{n}\right)^{\alpha} \approx 1 + \frac{\alpha}{n} \quad \text{for large} \quad n. \tag{2.51}$$

Thus the series converges if $\alpha > 1$, and diverges if $\alpha \leq 1$.

- Consider the series

  $$\sum_{n=1}^{\infty}\frac{1}{(\ln n)^{\alpha}}. \tag{2.52}$$

  Let's use Raabe's test:

  $$\left(\frac{\ln(n+1)}{\ln n}\right)^{\alpha}=\left(\frac{\ln n+\ln(1+1/n)}{\ln n}\right)^{\alpha}\approx 1+\frac{\alpha}{n\ln n}. \tag{2.53}$$

  Because

  $$n\left(\frac{\alpha}{n\ln n}\right)=\frac{\alpha}{\ln n}\to 0\quad\text{as}\quad n\to\infty, \tag{2.54}$$

  we conclude that the series is divergent.

- To test for convergence of

  $$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{\alpha}}, \tag{2.55}$$

  let us use the integral test:

  $$\begin{aligned}\int_{2}^{\infty}\frac{dx}{x(\ln x)^{\alpha}} &= \int_{\ln 2}^{\infty}\frac{d(\ln x)}{(\ln x)^{\alpha}}\\ &= \begin{cases}\frac{1}{1-\alpha}\frac{1}{(\ln x)^{\alpha-1}}\Big|_{x=2}^{\infty}, & \alpha\neq 1,\\ \ln(\ln x)\Big|_{x=2}^{\infty}, & \alpha=1\end{cases}\\ &= \begin{cases}\text{finite} & \alpha>1,\\ \infty & \alpha\leq 1.\end{cases}\end{aligned} \tag{2.56}$$

  Thus the series converges if $\alpha>1$ and diverges for other real α.