

Chapter 2

Infinite Series

2.1 Sequences

A *sequence* of complex numbers $\{z_n\}_{n=1}^{\infty}$ is a countably infinite set of numbers,

$$z_1, z_2, z_3, \dots, z_n, \dots \quad (2.1)$$

That is, for every positive integer k , there is a number, the k th term of the sequence, z_k , in the set $\{z_n\}_{n=1}^{\infty}$. Mathematically, a sequence is a complex-valued function defined on the positive integers.

We say that the sequence possesses a *limit* l ,

$$\lim_{n \rightarrow \infty} z_n = l, \quad \text{or} \quad z_n \rightarrow l \quad \text{as} \quad n \rightarrow \infty, \quad (2.2)$$

if, for every $\epsilon > 0$, no matter how small, there exists a number N for which

$$|z_n - l| < \epsilon \quad \text{for all} \quad n > N. \quad (2.3)$$

(The number N will depend on ϵ .) That is, $\{z_n\}_{n=N+1}^{\infty}$ all lie within a circle of radius ϵ centered on the point l in the complex plane.

A necessary and sufficient condition for a sequence $\{z_n\}_{n=1}^{\infty}$ to converge to a limit is *Cauchy's criterion*: A sequence $\{z_n\}_{n=1}^{\infty}$ possesses a limit if and only if for every $\epsilon > 0$, no matter how small, it is possible to find a number N such that

$$|z_n - z_m| < \epsilon \quad \text{for all} \quad n, m > N. \quad (2.4)$$

(Note that the difference $|n - m|$ may be arbitrarily large.) Thus, all elements of the sequence $\{z_n\}_{n=N+1}^{\infty}$ lie within a disk of radius ϵ . Briefly, we say that the Cauchy condition is

$$|z_n - z_m| \rightarrow 0 \quad \text{for all} \quad n, m \quad \text{sufficiently large.} \quad (2.5)$$

Sequences having this property are called *Cauchy sequences*. Every Cauchy sequence of complex numbers possesses a limit (which is, of course, a complex number)—this property means that the complex numbers form a *complete space*.

2.2 Series

Suppose we have a sequence $\{a_k\}_{k=1}^{\infty}$ from which we construct the finite sums

$$s_n = \sum_{k=1}^n a_k, \quad n = 1, 2, 3, \dots \quad (2.6)$$

The set of all these sums, $\{s_n\}_{n=1}^{\infty}$, itself forms a sequence. If this latter sequence has a limit S ,

$$s_n \rightarrow S \quad \text{as } n \rightarrow \infty, \quad (2.7)$$

then we say that the *infinite series*

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \quad (2.8)$$

possesses the limit S (or converges to S),

$$\sum_{k=1}^{\infty} a_k = S. \quad (2.9)$$

By the Cauchy criterion, this will be true if and only if

$$\left| \sum_{k=m}^n a_k \right| < \epsilon \quad (2.10)$$

for any fixed $\epsilon > 0$, whenever $n \geq m > N$, N a number depending on ϵ .

Obviously, a necessary condition for

$$\sum_{k=1}^{\infty} a_k \quad (2.11)$$

to converge is for $a_k \rightarrow 0$ as $k \rightarrow \infty$. However, this is not sufficient, as the following example shows.

2.3 Examples

2.3.1 Harmonic series

Consider the sum of the reciprocals of the integers,

$$1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}. \quad (2.12)$$

Note that if the n th term of the series is denoted $a_n = 1/n$, we have for the sum of n adjacent terms

$$a_{n+1} + \dots + a_{2n} = \frac{1}{n+1} + \dots + \frac{1}{2n} > n \left(\frac{1}{2n} \right) = \frac{1}{2}, \quad (2.13)$$

no matter how large n is. This violates Cauchy's criterion, so the harmonic series *diverges*.

2.3.2 Geometric series

Consider the series

$$\sum_{m=0}^{\infty} ar^m, \quad (2.14)$$

where a is a constant and $r \geq 0$. For $r \neq 1$, the n th partial sum is

$$s_n = \sum_{m=0}^n ar^m = a \frac{1-r^{n+1}}{1-r}, \quad (2.15)$$

so

$$S = \lim_{n \rightarrow \infty} s_n = \frac{a}{1-r} \quad \text{if } r < 1, \quad (2.16)$$

while the series diverges if $r \geq 1$.

2.4 Absolute and Conditional Convergence

Suppose we have a convergent series $\sum_{n=1}^{\infty} a_n$. If also $\sum_{n=1}^{\infty} |a_n|$ converges, we say that the original series converges *absolutely*. Otherwise, the original series is *conditionally convergent*. (That is, it converges because of sign alternations.) A sufficient condition for (at least) conditional convergence is provided by the following theorem due to Leibnitz:

If the terms of a series are of alternating sign and in addition their absolute values tend to zero, $|a_n| \rightarrow 0$, monotonically, i.e., $|a_n| > |a_{n+1}|$ for sufficiently large n , then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges.} \quad (2.17)$$

In absolutely convergent series one can rearrange the terms without affecting the value of the sum. With conditionally convergent series, one cannot rearrange terms; in fact, such rearrangements can make a conditionally convergent series converge to any desired value, or to diverge!

2.4.1 Example

Consider the conditionally convergent series formed from the divergent harmonic series by alternating every other sign:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2, \quad (2.18)$$

which converges to the natural logarithm of 2. Multiply this equation term by term by $1/2$:

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \dots = \frac{1}{2} \ln 2. \quad (2.19)$$

Add these two series:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots = \frac{3}{2} \ln 2. \quad (2.20)$$

Since the reciprocal of each integer occurs exactly once in the last series, we would be tempted to rearrange the series to obtain

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2, \quad (2.21)$$

which is identical to the original series. There is an obvious contradiction here! In order to obtain the rearrangement (2.21), we have to go further and further out in the series (2.20), which apparently is not permissible.

2.4.2 A Theorem About Absolutely Convergent Series

Not only can absolutely convergent series be rearranged without changing their value, but they can be multiplied together term by term: If two series

$$S = \sum_{i=1}^{\infty} u_i, \quad (2.22a)$$

$$T = \sum_{i=1}^{\infty} v_i \quad (2.22b)$$

are both *absolutely* convergent, the series

$$P = \sum_{\substack{i=1 \\ j=1}}^{\infty} u_i v_j \quad (2.23)$$

formed from the product of their terms written in any order, is absolutely convergent, and has a value equal to the product of the individual series,

$$P = ST. \quad (2.24)$$

2.5 Convergence Tests

The following tests can determine whether a given series is absolutely convergent or not.

2.5.1 Comparison test

If $b_n > 0$ for all n and $\sum_{n=1}^{\infty} b_n$ is convergent, and if $|a_n| \leq b_n$ for all n , then

$$\sum_{n=1}^{\infty} a_n \quad \text{is absolutely convergent.} \quad (2.25a)$$

Also, if $|a_n| \geq b_n > 0$ for all n , and $\sum_{n=1}^{\infty} b_n$ diverges, then

$$\sum_{n=1}^{\infty} a_n \text{ is not absolutely convergent.} \quad (2.25b)$$

2.5.2 Root test

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if from a certain term onward

$$\sqrt[n]{|a_n|} \leq q < 1, \quad (2.26)$$

where $q \geq 0$ is independent of n .

Proof: If the inequality holds, $|a_n| \leq q^n$. But $\sum_{n=1}^{\infty} q^n$ converges for $q < 1$, it being the geometric series, so by 2.5.1, $\sum_{n=1}^{\infty} |a_n|$ converges.

2.5.3 Ratio test

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if from a certain term onward

$$\left| \frac{a_{n+1}}{a_n} \right| \leq q < 1, \quad (2.27)$$

where $q \geq 0$ is independent of n .

Proof: Without loss of generality, we may assume the inequality holds for all n ; otherwise, we renumber the $\{a_n\}$ sequence so that 1 labels the first term for which the inequality (2.27) holds. Then

$$\left| \frac{a_n}{a_1} \right| = \left| \frac{a_n}{a_{n-1}} \right| \left| \frac{a_{n-1}}{a_{n-2}} \right| \left| \frac{a_{n-2}}{a_{n-3}} \right| \cdots \left| \frac{a_2}{a_1} \right| \leq q^{n-1}. \quad (2.28)$$

Convergence is again assured by comparison with the geometric series. (Whether these tests are satisfied by the first few terms of a series is immaterial, since a finite number of terms of an infinite series has no effect on the convergence.)

Example

When does $\sum_{n=1}^{\infty} nq^n$ converge? If we use the root test, we examine

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = |q| \lim_{n \rightarrow \infty} \sqrt[n]{n} = |q|,^1 \quad (2.29a)$$

while if we use the ratio test, we look at

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |q| \lim_{n \rightarrow \infty} \frac{n+1}{n} = |q|. \quad (2.29b)$$

In either case, we see that the series is absolutely convergent if $|q| < 1$, and divergent otherwise.

¹Because $\ln \sqrt[n]{n} = \frac{1}{n} \ln n$, which tends to zero as $n \rightarrow \infty$, $\sqrt[n]{n} \rightarrow 1$.

The following are refinements of the ratio test, which fails (that is, fails to reveal whether the tested series converges or not) when

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1. \quad (2.30)$$

For example, this indeterminate limit results for the case $a_n = 1/n$, which yields a divergent series, but also for $a_n = 1/(n \ln^2 n)$, which corresponds to a convergent sum (see Sec. 2.5.8).

2.5.4 Kummer's test

Choose a sequence of positive constants b_n . If

$$b_n \left| \frac{a_n}{a_{n+1}} \right| - b_{n+1} \geq C > 0, \quad (2.31)$$

for all $n \geq N$, where N and C are fixed numbers, then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges absolutely.} \quad (2.32)$$

On the other hand, if

$$b_n \left| \frac{a_n}{a_{n+1}} \right| - b_{n+1} \leq 0, \quad (2.33)$$

and

$$\sum_{n=1}^{\infty} b_n^{-1} \quad \text{diverges,} \quad (2.34)$$

then

$$\sum_{n=1}^{\infty} |a_n| \quad \text{diverges.} \quad (2.35)$$

Proof: If the inequality (2.31) holds, take $l \geq N$, so that

$$C|a_{l+1}| \leq b_l|a_l| - b_{l+1}|a_{l+1}|. \quad (2.36)$$

So we have the inequality

$$\sum_{l=N+1}^n |a_l| \leq \frac{b_N|a_N|}{C} - \frac{b_n|a_n|}{C} \leq \frac{b_N|a_N|}{C}. \quad (2.37)$$

Hence, the n th partial sum, for $n > N$, is

$$s_n = \sum_{i=1}^n |a_i| \leq \sum_{i=1}^N |a_i| + \frac{b_N|a_N|}{C}. \quad (2.38)$$

The right-hand side of this inequality is a constant, independent of n . Therefore, the positive sequence of increasing terms $\{s_n\}$ is bounded above, and consequently possesses a limit. The series is absolutely convergent.

If the inequality (2.33) holds,

$$|a_n| \geq \frac{|a_N|b_N}{b_n}, \quad n > N, \quad (2.39)$$

so since $\sum_{n=1}^{\infty} b_n^{-1}$ diverges, so does $\sum_{n=1}^{\infty} |a_n|$.

2.5.5 Raabe's test

Raabe's criterion for absolute convergence is

$$n \left(\left| \frac{a_n}{a_{n+1}} \right| - 1 \right) \geq K > 1, \quad (2.40)$$

for all $n \geq N$, where N and K are fixed. And if

$$n \left(\left| \frac{a_n}{a_{n+1}} \right| - 1 \right) \leq 1, \quad (2.41)$$

then

$$\sum_{n=1}^{\infty} |a_n| \text{ diverges.} \quad (2.42)$$

Proof: In Kummer's test put $b_n = n$.

2.5.6 Gauss' test

If

$$\left| \frac{a_n}{a_{n+1}} \right| = 1 + \frac{h}{n} + \frac{B(n)}{n^2}, \quad (2.43)$$

where h is a constant and the function $B(n)$ is bounded as $n \rightarrow \infty$, then $\sum_{n=1}^{\infty} |a_n|$ converges for $h > 1$ and diverges for $h \leq 1$.

Proof: For $h \neq 1$ we can use Raabe's test:

$$\lim_{n \rightarrow \infty} n \left(\frac{h}{n} + \frac{B(n)}{n^2} \right) = h. \quad (2.44)$$

For $h = 1$, Raabe's test is indeterminate. In that case use Kummer's test with $b_n = n \ln n$: for large n ,

$$\begin{aligned} & n \ln n \left(1 + \frac{h}{n} + \frac{B(n)}{n^2} \right) - (n+1) \ln(n+1) \\ & \approx n \ln n \left(1 + \frac{h}{n} + \frac{B(n)}{n^2} \right) - (n+1) \left(\ln n + \frac{1}{n} \right) \\ & \approx \left(h + \frac{B(n)}{n} \right) \ln n - \ln n - 1 \\ & \approx (h-1) \ln n - 1 < 0, \quad \text{if } h \leq 1. \end{aligned} \quad (2.45)$$

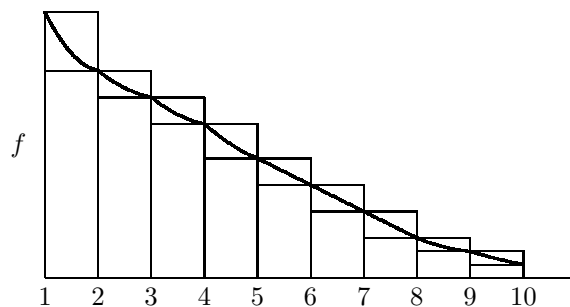


Figure 2.1: Bounds on a monotone series provided by an integral.

Because

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges} \quad (2.46)$$

(see homework), the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

2.5.7 Integral test

If $f(x)$ is a continuous, monotonically decreasing real function of x such that

$$f(n) = |a_n|, \quad (2.47)$$

then

$$\sum_{n=1}^{\infty} |a_n| \text{ converges if } \int_1^{\infty} dx f(x) < \infty, \quad (2.48)$$

and diverges otherwise.

Proof: It is geometrically obvious that

$$\int_1^{\infty} dx f(x) < \sum_{n=1}^{\infty} f(n) < \int_1^{\infty} dx f(x) + f(1), \quad (2.49)$$

for this follows merely from the geometrical meaning of the integral as the area under the curve of the function. See Fig. 2.1.

2.5.8 Examples

- The Riemann zeta function is defined by the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} = \zeta(\alpha). \quad (2.50)$$

We can test for convergence using Gauss' test, by examining

$$\left(\frac{n+1}{n}\right)^{\alpha} \approx 1 + \frac{\alpha}{n} \text{ for large } n. \quad (2.51)$$

Thus the series converges if $\alpha > 1$, and diverges if $\alpha \leq 1$.

- Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\alpha}}. \quad (2.52)$$

Let's use Raabe's test:

$$\left(\frac{\ln(n+1)}{\ln n}\right)^{\alpha} = \left(\frac{\ln n + \ln(1+1/n)}{\ln n}\right)^{\alpha} \approx 1 + \frac{\alpha}{n \ln n}. \quad (2.53)$$

Because

$$n \left(\frac{\alpha}{n \ln n}\right) = \frac{\alpha}{\ln n} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2.54)$$

we conclude that the series is divergent.

- To test for convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\alpha}}, \quad (2.55)$$

let us use the integral test:

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x(\ln x)^{\alpha}} &= \int_{\ln 2}^{\infty} \frac{d(\ln x)}{(\ln x)^{\alpha}} \\ &= \begin{cases} \frac{1}{1-\alpha} \frac{1}{(\ln x)^{\alpha-1}} \Big|_{x=2}^{\infty}, & \alpha \neq 1, \\ \ln(\ln x) \Big|_{x=2}^{\infty}, & \alpha = 1 \end{cases} \\ &= \begin{cases} \text{finite} & \alpha > 1, \\ \infty & \alpha \leq 1. \end{cases} \end{aligned} \quad (2.56)$$

Thus the series converges if $\alpha > 1$ and diverges for other real α .

2.6 Series of Functions

2.6.1 Continuity

A (complex-valued) function $f(z)$ of a complex variable is *continuous* at z_0 if

$$f(z) \rightarrow f(z_0) \quad \text{as } z \rightarrow z_0 \quad (2.57)$$

from any direction. That is, given $\epsilon > 0$ we may find a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever } |z - z_0| < \delta. \quad (2.58)$$

In other words, z lies within a circle of radius δ around z_0 .

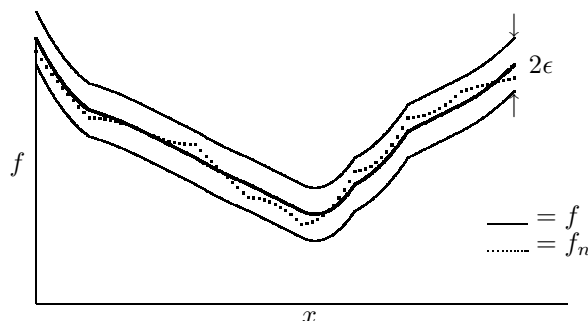


Figure 2.2: Uniform convergence of the partial sum $f_n(x)$ to the limit $f(x)$. For all x , $f_n(x)$ is within a band of width 2ϵ about $f(x)$.

2.6.2 Uniform Convergence

Consider the infinite series

$$f(z) = \sum_{i=1}^{\infty} g_i(z) \quad (2.59)$$

constructed from the sequence of functions $\{g_i\}_{i=1}^{\infty}$. The condition that this series converge is expressed in terms of the partial sums,

$$f_n(z) = \sum_{i=1}^n g_i(z) \quad (2.60)$$

thusly: given $\epsilon > 0$ we can find an integer N so that for $n > N$

$$|f_{n+p}(z) - f_n(z)| < \epsilon \quad \text{for all } p > 0. \quad (2.61)$$

This is Cauchy's criterion. In general the N required for this to occur will depend on the point z . If, however, Eq. (2.61) holds for all z if $n > N$ independent of z , we say that the series *converges uniformly* throughout the region of interest. Equivalently, there exists a function $f(z)$ such that

$$|f(z) - f_n(z)| < \epsilon \quad \text{for all } n > N, \quad N \text{ independent of } z. \quad (2.62)$$

That is, the partial sum f_n is everywhere uniformly close to f , the limiting function. This situation is illustrated in Fig. 2.2 for a real function of a real variable.

Contrast absolute and uniform convergence through the following examples. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n + z^2} \quad (2.63)$$

is only conditionally convergent, because asymptotically the terms become $(-1)^n/n$. On the other hand, for real z it is uniformly convergent because

$$\left| \sum_{n=N+1}^{N+p} \frac{(-1)^n}{n + z^2} \right| < \frac{1}{N + z^2} \leq \frac{1}{N}, \quad (2.64)$$

which is the Cauchy criterion with $\epsilon = 1/N$.

In contrast, consider, for real z , the series

$$S(z) = \sum_{n=0}^{\infty} \frac{z^2}{(1+z^2)^n} \quad (2.65)$$

which converges absolutely. For $z = 0$, $S(0) = 0$; and for $z \neq 0$,

$$S(z) = z^2 \sum_{n=0}^{\infty} \frac{1}{(1+z^2)^n} = \frac{z^2}{1 - \frac{1}{1+z^2}} = 1 + z^2. \quad (2.66)$$

Thus $S(z)$ is discontinuous at $z = 0$. The following theorem shows that this series cannot be uniformly convergent there.

Theorem

If a series of continuous functions of z is uniformly convergent for all values of z in a given closed domain, the sum is continuous throughout the domain.

Proof: Let

$$f_n(z) = \sum_{i=1}^n g_i(z). \quad (2.67)$$

Since

$$f_n(z) \rightarrow f(z) \quad \text{uniformly,} \quad (2.68)$$

we can find, for any $\epsilon > 0$, a value of n such that

$$|f_n(z) - f(z)| < \epsilon \quad \text{for all } z \quad (2.69)$$

throughout the domain. Then

$$\begin{aligned} |f(z) - f(z')| &= |f(z) - f_n(z) + f_n(z) - f(z') + f_n(z') - f_n(z')| \\ &\leq |f(z) - f_n(z)| + |f(z') - f_n(z')| + |f_n(z) - f_n(z')|. \end{aligned} \quad (2.70)$$

Since the f_n 's are continuous, we can find a δ for any given ϵ such that

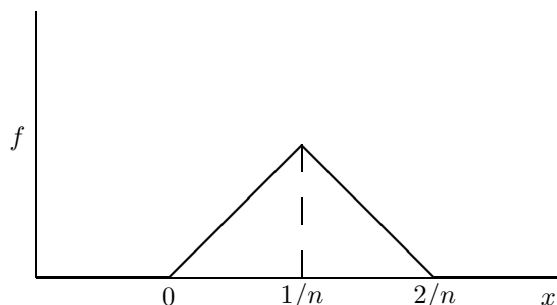
$$|f_n(z) - f_n(z')| < \epsilon \quad \text{whenever } |z - z'| < \delta. \quad (2.71)$$

Therefore,

$$|f(z) - f(z')| < 3\epsilon \quad \text{whenever } |z - z'| < \delta. \quad (2.72)$$

QED.

Even if the limit function is continuous, convergence to it need not be uniform, as the following example shows:

Figure 2.3: Sketch of the function $f_n(x)$ given by Eq. 2.73).**Example**

Consider the sequence of continuous functions,

$$f_n(x) = \begin{cases} nx, & 0 \leq x \leq 1/n, \\ (2/n - x)n, & 1/n \leq x \leq 2/n, \\ 0, & \text{otherwise.} \end{cases} \quad (2.73)$$

This function is sketched in Fig. 2.3. Note that the maximum of the function $f_n(x)$ is 1. On the other hand, for all x ,

$$\lim_{n \rightarrow \infty} f_n(x) = 0, \quad (2.74)$$

which is certainly a continuous limit function. But the convergence to this limit is not uniform, for there is always a point, $x = 1/n$, for which

$$\left| 0 - f_n\left(\frac{1}{n}\right) \right| = 1 \quad (2.75)$$

no matter how large n is. So the convergence is nonuniform.

Properties of Uniformly Convergent Series

Consider the series of functions of a real variable,

$$f(x) = \sum_{n=1}^{\infty} g_n(x). \quad (2.76)$$

1. If the g_n are continuous, we can integrate term by term if $\sum_n g_n$ is uniformly convergent over the domain of integration:

$$\int_a^b dx f(x) = \sum_{n=1}^{\infty} \int_a^b dx g_n(x). \quad (2.77)$$

2. If the g_n and $g'_n = \frac{d}{dx}g_n$ are continuous, and $\sum_n g'_n$ is uniformly convergent, then we can differentiate term by term:

$$f'(x) = \sum_{n=1}^{\infty} g'_n(x). \quad (2.78)$$

Condition for Uniform Convergence

The following condition is sufficient, but not necessary, to ensure that a series is uniformly convergent.

If $|g_n(z)| < a_n$, where $\{a_n\}$ is a sequence of constants such that $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} g_n(z)$ converges uniformly and absolutely.

Proof: The hypothesis implies

$$\left| \sum_{n=N}^{N+p} g_n(z) \right| < \sum_{n=N}^{N+p} a_n, \quad (2.79)$$

so that if N is chosen so that $\sum_{n=N}^{N+p} a_n < \epsilon$, then

$$\left| \sum_{n=N}^{N+p} g_n(z) \right| < \epsilon \quad \forall z. \quad (2.80)$$

2.7 Power Series

By a *power series*, we mean a series of the form,

$$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \dots, \quad (2.81)$$

where the c_n s form a sequence of complex constants, and z is a complex variable.

If a power series converges for one point, $z = z_0$, it converges *uniformly and absolutely* for all z satisfying

$$|z| \leq \eta, \quad (2.82)$$

where η is any positive number less than $|z_0|$.

Proof: Since $\sum_{n=0}^{\infty} c_n z_0^n$ converges, it must be true that the terms are bounded,

$$|c_n z_0^n| < M, \quad (2.83)$$

where M is independent of n (but not of $|z_0|$). Hence if Eq. (2.82) is satisfied,

$$\sum_{n=0}^{\infty} |c_n z^n| \leq \sum_{n=0}^{\infty} |c_n| \eta^n < M \sum_{n=0}^{\infty} \left(\frac{\eta}{|z_0|} \right)^n < \infty, \quad (2.84)$$

since $\eta/|z_0| < 1$. This proves absolute convergence. Uniform convergence follows from the theorem above.

2.7.1 Radius of Convergence

Use the root test to determine where the power series

$$\sum_{n=0}^{\infty} c_n z^n \quad (2.85)$$

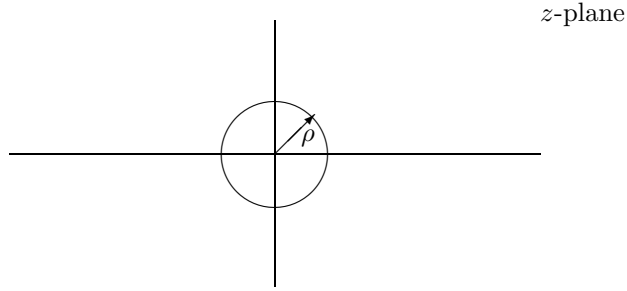


Figure 2.4: Circle of convergence of a power series. The series (2.85) converges inside the circle, and diverges outside. The radius of convergence ρ is given by Eq. (2.87).

converges. That test says if

$$\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} |z| < 1, \quad \text{the series converges,} \quad (2.86a)$$

while if

$$\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} |z| > 1, \quad \text{the series diverges.} \quad (2.86b)$$

Therefore, the power series converges within a *circle of convergence* of radius ρ , the *radius of convergence*, where

$$\rho = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|}}, \quad (2.87)$$

and diverges outside that circle, as shown in Fig. 2.4. More detailed examination is required to determine whether or not the series converges *on* the circle of convergence.

2.7.2 Properties of Power Series Within the Circle of Convergence

1. The function defined by the power series is continuous. [This follows from the theorem in Sec. 2.6.2.]
2. It may be differentiated or integrated term by term. [This follows from the theorem above, together with the fact that if $\sum_{n=0}^{\infty} c_n z^n$ converges, so does $\sum_{n=0}^{\infty} n c_n z^{n-1}$, by the ratio test,

$$\left| \frac{(n+1)c_{n+1}z^n}{n c_n z^{n-1}} \right| = \frac{n+1}{n} \left| \frac{c_{n+1}}{c_n} \right| |z|. \quad (2.88)$$

Now if z lies within the circle of convergence,

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| |z| < 1. \quad (2.89)$$

Since $\lim_{n \rightarrow \infty} (n+1)/n = 1$, convergence of the differentiated series is assured.]

3. Two such power series may be multiplied together term by term, within the smaller of the two circles of convergence. [This follows from the theorem in Sec. 2.4.2.]
4. The power series is unique. [It suffices to show that if

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = 0 \quad \forall z, \quad c_n = 0 \quad \forall n. \quad (2.90)$$

Indeed,

$$f(0) = c_0 = 0, \quad (2.91a)$$

$$f'(0) = c_1 = 0, \quad (2.91b)$$

...

$$f^{(n)}(0) = n! c_n = 0. \quad (2.91c)$$

2.7.3 Taylor Expansion

The *Taylor expansion* for a real function of a real variable is obtained from the above argument. If we write a function as a power series,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad (2.92)$$

then

$$c_n = \frac{1}{n!} f^{(n)}(0). \quad (2.93)$$

Hence, the power series is the Taylor series of the function it represents,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n. \quad (2.94)$$

2.7.4 Hypergeometric Function

The hypergeometric function F is defined by the power series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \quad (2.95)$$

Here the coefficients are defined in terms of the Pochhammer symbol,

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (2.96)$$

To determine convergence, we examine

$$\begin{aligned}
 \frac{A_n z^n}{A_{n+1} z^{n+1}} &= \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{\Gamma(c+n+1)}{\Gamma(a+n+1)\Gamma(b+n+1)} \frac{(n+1)!}{n!} \frac{z^n}{z^{n+1}} \\
 &= \frac{\left(1 + \frac{c}{n}\right) \left(1 + \frac{1}{n}\right)}{\left(1 + \frac{a}{n}\right) \left(1 + \frac{b}{n}\right)} \frac{1}{z} \\
 &= \frac{1}{z} \left[1 + \frac{1}{n}(c+1-a-b) + O\left(\frac{1}{n^2}\right) \right], \tag{2.97}
 \end{aligned}$$

where $O(1/n^2)$ means that the next term goes to zero as $n \rightarrow \infty$ at least as fast as $1/n^2$. According to the ratio test, the radius of convergence of this series is $|z| = 1$; that is, the series diverges for $|z| > 1$, and converges uniformly and absolutely for any z such that $|z| \leq \eta < 1$. The remaining question is what happens on the circle of convergence, $|z| = 1$. According to Gauss' test, the series is then absolutely convergent if $c > a + b$ [if the constants are complex, if $\operatorname{Re}(c - a - b) > 0$]. For the point $z = 1$ the series is certainly divergent if this condition is not satisfied; however, if $-1 < \operatorname{Re}(c - a - b) \leq 0$ the series is conditionally convergent on the unit circle except for the exceptional point $z = 1$. On the other hand, if $\operatorname{Re}(c - a - b) \leq -1$ the series is divergent on the unit circle because the terms in the series increase in magnitude.