Chapter 1

Review of Complex Numbers

Complex numbers are defined in terms of the *imaginary unit*, i, having the property

$$i^2 = -1. \tag{1.1}$$

A general *complex number* has the form

$$z = x + iy, \tag{1.2}$$

where x, y are real numbers. We also often write

$$z = \operatorname{Re} z + i \operatorname{Im} z, \tag{1.3}$$

where $\operatorname{Re} z$ is the "real part of z," and $\operatorname{Im} z$ is the "imaginary part of z." Complex numbers are added and multiplied just like real numbers: If

$$z_1 = x_1 + iy_1, \tag{1.4a}$$

$$z_2 = x_2 + iy_2, \tag{1.4b}$$

then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \tag{1.5a}$$

$$\begin{aligned} z_1 z_2 &= x_1 x_2 + iy_1 x_2 + ix_1 y_2 + i^2 y_1 y_2 \\ &= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1). \end{aligned} \tag{1.5b}$$

The *complex conjugate* of a number is obtained by reversing the sign of i: If $z = x + iy$, we define the complex conjugate of z by

$$z^* = x - iy. \tag{1.6}$$

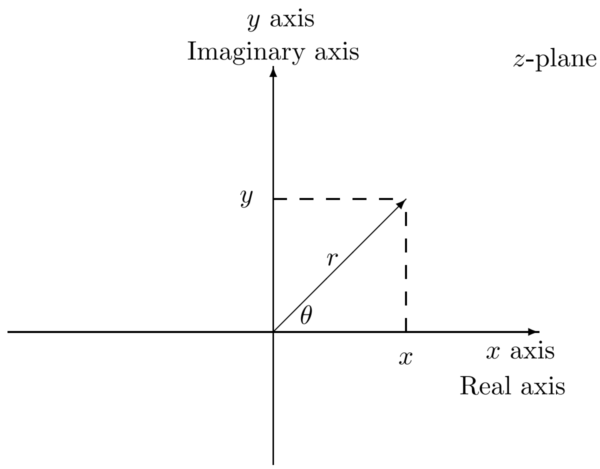

Figure 1.1: Geometrical interpretation of a complex number $z = x + iy$.

(Sometimes the notation $\bar{z}$ is used for the complex conjugate of z.) Note that

$$\text{Re } z = \frac{z + z^*}{2}, \tag{1.7a}$$

$$\text{Im } z = \frac{z - z^*}{2i}. \tag{1.7b}$$

Note also that

$$zz^* = x^2 + y^2 \tag{1.8}$$

is purely real and non-negative, so we define the *modulus*, or *magnitude*, or *absolute value* of z by

$$|z| = \sqrt{zz^*} = \sqrt{(\text{Re } z)^2 + (\text{Im } z)^2}, \tag{1.9}$$

where the positive square root is implied.

We give a simple geometrical interpretation to complex numbers, by thinking of them as two-dimensional vectors, as sketched in Fig. 1.1. Here the length of the vector is the magnitude of the complex number,

$$r = |z|, \tag{1.10}$$

and the angle the vector makes with the real axis is θ, where

$$\tan\theta = y/x; \tag{1.11}$$

the quadrant θ lies in is determined by the sign of x and y. We call

$$\theta = \arg z \tag{1.12}$$

the *argument* or *phase* of z. The above geometrical picture is sometimes called an *Argand diagram*.

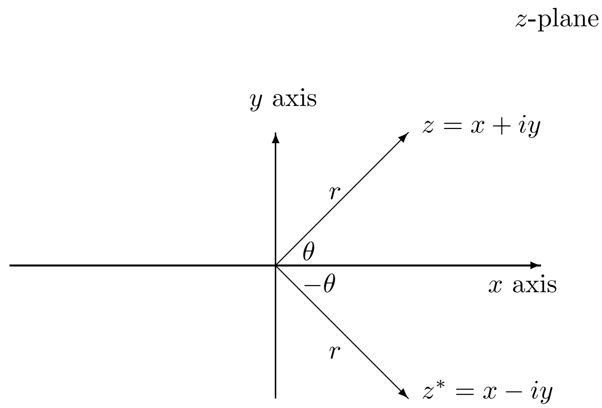

Figure 1.2: Geometrical interpretation of complex conjugation.

There is an arbitrariness in the choice of the argument θ of a complex number z, for one can always add an arbitrary multiple of 2π to θ without changing z,

$$\theta \to \theta + 2\pi n, \quad n \text{ an integer}, \quad z \to z. \tag{1.13}$$

It is often convenient to define a single-valued argument function $\arg z$. By convention, the *principal value* of $\arg z$ is that phase angle which satisfies the inequality

$$-\pi < \arg z \leq \pi. \tag{1.14}$$

(Note that radian measure is always employed.) For every z there is a unique $\arg z$ lying in this range.

The geometrical significance of complex conjugation is shown in Fig. 1.2. Complex conjugation corresponds to reflection in the x-axis.

From the Argand diagram we can write down the “polar representation” of a complex number,

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta), \end{aligned} \tag{1.15}$$

so if we have two complex numbers,

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \tag{1.16a}$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2), \tag{1.16b}$$

the product is

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ &\qquad + i\,[\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1]\} \\ &= r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]. \end{aligned} \tag{1.17}$$

That is, the moduli of the complex numbers multiply,

$$|z_1 z_2| = |z_1||z_2|, \tag{1.18a}$$

while the arguments add,

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{1.18b}$$

The latter statement is to be understood as modulo 2π, i.e., equality up to the addition of an arbitrary integer multiple of 2π. In particular, note that

$$\left|\frac{1}{z} z\right| = \left|\frac{1}{z}\right| |z| = 1, \tag{1.19a}$$

while

$$0 = \arg\left(\frac{1}{z} z\right) = \arg\frac{1}{z} + \arg z, \tag{1.19b}$$

implying that

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}, \tag{1.20a}$$

$$\arg\frac{1}{z} = -\arg z. \tag{1.20b}$$

1.1 De Moivre's Theorem

From the above, if we choose a unit vector,

$$z = \cos\theta + i\sin\theta, \tag{1.21}$$

successive powers follow a simple pattern:

$$z^2 = \cos 2\theta + i\sin 2\theta, \tag{1.22a}$$

$$z^3 = \cos 3\theta + i\sin 3\theta, \tag{1.22b}$$

$$\ldots,$$

$$z^n = \cos n\theta + i\sin n\theta, \tag{1.22c}$$

or

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{1.23}$$

where n is a positive integer. This is called *De Moivre's theorem.*

1.2 Roots

Suppose we wish to find all the nth roots of unity, that is, all solutions to the equation

$$z^n = 1, \tag{1.24}$$

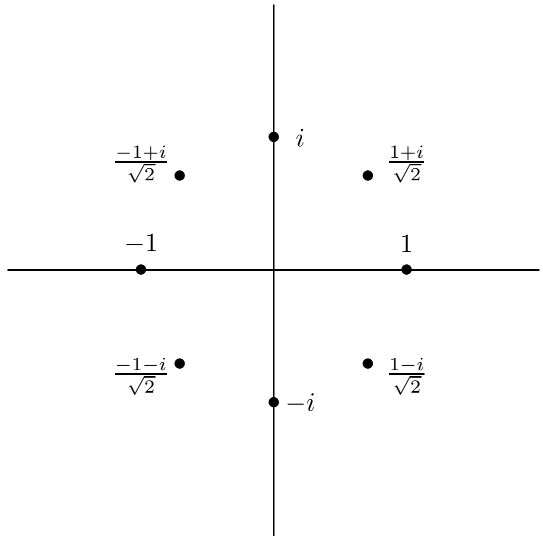

Figure 1.3: The eight 8th roots of unity.

where n is a positive integer. If we take the polar form,

$$z = \rho(\cos\phi + i\sin\phi), \tag{1.25}$$

this means

$$\rho^n(\cos n\phi + i\sin n\phi) = 1, \tag{1.26}$$

which implies

$$\rho = 1, \tag{1.27a}$$

$$n\phi = 2\pi k, \tag{1.27b}$$

where k is any integer. Thus the nth root of unity has the form

$$z = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}. \tag{1.28}$$

These are distinct for

$$k = 0,\, 1,\, 2,\, \ldots,\, n-1; \tag{1.29}$$

outside of these values of k, the roots repeat. Thus there are n distinct nth roots of unity. For example, for $n = 8$, the roots are as shown in Fig. 1.3, in the complex plane.