

Chapter 1

Review of Complex Numbers

Complex numbers are defined in terms of the *imaginary unit*, i , having the property

$$i^2 = -1. \quad (1.1)$$

A general *complex number* has the form

$$z = x + iy, \quad (1.2)$$

where x, y are real numbers. We also often write

$$z = \operatorname{Re} z + i\operatorname{Im} z, \quad (1.3)$$

where $\operatorname{Re} z$ is the “real part of z ,” and $\operatorname{Im} z$ is the “imaginary part of z .” Complex numbers are added and multiplied just like real numbers: If

$$z_1 = x_1 + iy_1, \quad (1.4a)$$

$$z_2 = x_2 + iy_2, \quad (1.4b)$$

then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \quad (1.5a)$$

$$\begin{aligned} z_1 z_2 &= x_1 x_2 + iy_1 x_2 + ix_1 y_2 + i^2 y_1 y_2 \\ &= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1). \end{aligned} \quad (1.5b)$$

The *complex conjugate* of a number is obtained by reversing the sign of i : If $z = x + iy$, we define the complex conjugate of z by

$$z^* = x - iy. \quad (1.6)$$

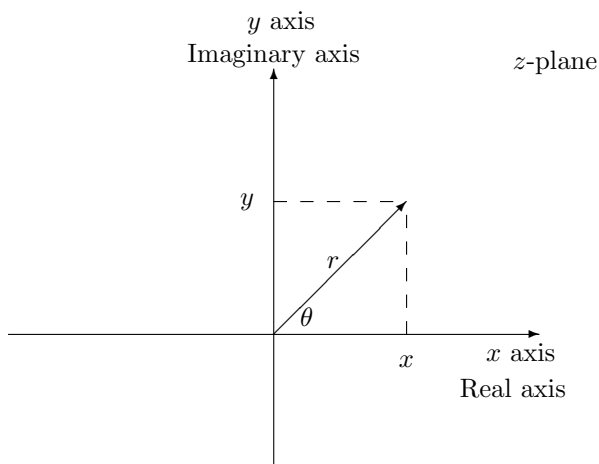


Figure 1.1: Geometrical interpretation of a complex number $z = x + iy$.

(Sometimes the notation \bar{z} is used for the complex conjugate of z .) Note that

$$\operatorname{Re} z = \frac{z + z^*}{2}, \quad (1.7a)$$

$$\operatorname{Im} z = \frac{z - z^*}{2i}. \quad (1.7b)$$

Note also that

$$zz^* = x^2 + y^2 \quad (1.8)$$

is purely real and non-negative, so we define the *modulus*, or *magnitude*, or *absolute value* of z by

$$|z| = \sqrt{zz^*} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}, \quad (1.9)$$

where the positive square root is implied.

We give a simple geometrical interpretation to complex numbers, by thinking of them as two-dimensional vectors, as sketched in Fig. 1.1. Here the length of the vector is the magnitude of the complex number,

$$r = |z|, \quad (1.10)$$

and the angle the vector makes with the real axis is θ , where

$$\tan \theta = y/x; \quad (1.11)$$

the quadrant θ lies in is determined by the sign of x and y . We call

$$\theta = \arg z \quad (1.12)$$

the *argument* or *phase* of z . The above geometrical picture is sometimes called an *Argand diagram*.

z-plane

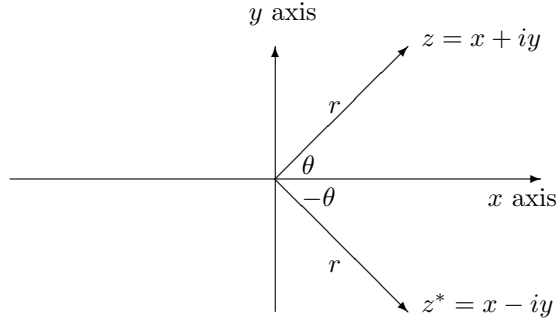


Figure 1.2: Geometrical interpretation of complex conjugation.

There is an arbitrariness in the choice of the argument θ of a complex number z , for one can always add an arbitrary multiple of 2π to θ without changing z ,

$$\theta \rightarrow \theta + 2\pi n, \quad n \text{ an integer}, \quad z \rightarrow z. \quad (1.13)$$

It is often convenient to define a single-valued argument function $\arg z$. By convention, the *principal value* of $\arg z$ is that phase angle which satisfies the inequality

$$-\pi < \arg z \leq \pi. \quad (1.14)$$

(Note that radian measure is always employed.) For every z there is a unique $\arg z$ lying in this range.

The geometrical significance of complex conjugation is shown in Fig. 1.2. Complex conjugation corresponds to reflection in the x -axis.

From the Argand diagram we can write down the “polar representation” of a complex number,

$$\begin{aligned} z &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta), \end{aligned} \quad (1.15)$$

so if we have two complex numbers,

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1), \quad (1.16a)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2), \quad (1.16b)$$

the product is

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \{ \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \\ &\quad + i [\cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1] \} \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \end{aligned} \quad (1.17)$$

That is, the moduli of the complex numbers multiply,

$$|z_1 z_2| = |z_1| |z_2|, \quad (1.18a)$$

while the arguments add,

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \quad (1.18b)$$

The latter statement is to be understood as modulo 2π , i.e., equality up to the addition of an arbitrary integer multiple of 2π . In particular, note that

$$\left| \frac{1}{z} z \right| = \left| \frac{1}{z} \right| |z| = 1, \quad (1.19a)$$

while

$$0 = \arg \left(\frac{1}{z} z \right) = \arg \frac{1}{z} + \arg z, \quad (1.19b)$$

implying that

$$\left| \frac{1}{z} \right| = \frac{1}{|z|}, \quad (1.20a)$$

$$\arg \frac{1}{z} = -\arg z. \quad (1.20b)$$

1.1 De Moivre's Theorem

From the above, if we choose a unit vector,

$$z = \cos \theta + i \sin \theta, \quad (1.21)$$

successive powers follow a simple pattern:

$$z^2 = \cos 2\theta + i \sin 2\theta, \quad (1.22a)$$

$$z^3 = \cos 3\theta + i \sin 3\theta, \quad (1.22b)$$

...

$$z^n = \cos n\theta + i \sin n\theta, \quad (1.22c)$$

or

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \quad (1.23)$$

where n is a positive integer. This is called *De Moivre's theorem*.

1.2 Roots

Suppose we wish to find all the n th roots of unity, that is, all solutions to the equation

$$z^n = 1, \quad (1.24)$$

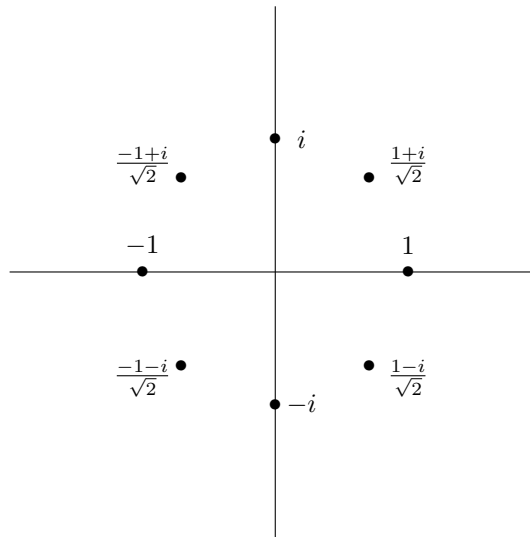


Figure 1.3: The eight 8th roots of unity.

where n is a positive integer. If we take the polar form,

$$z = \rho(\cos \phi + i \sin \phi), \quad (1.25)$$

this means

$$\rho^n(\cos n\phi + i \sin n\phi) = 1, \quad (1.26)$$

which implies

$$\rho = 1, \quad (1.27a)$$

$$n\phi = 2\pi k, \quad (1.27b)$$

where k is any integer. Thus the n th root of unity has the form

$$z = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}. \quad (1.28)$$

These are distinct for

$$k = 0, 1, 2, \dots, n-1; \quad (1.29)$$

outside of these values of k , the roots repeat. Thus there are n distinct n th roots of unity. For example, for $n = 8$, the roots are as shown in Fig. 1.3, in the complex plane.