Chapter 24

Study Guide

Chapter 12 discussed compatible observables, which can be simultaneously measured and specified. These are characterized by the fact that compatible physical properties are represented by Hermitian operators that commute with each other: If A and B are compatible,

$$[A, B] = 0. \tag{24.1}$$

A state of a physical system is characterized by specifying the values of a maximal number of compatible observables,

$$(A_i - a_i')|\{a_1', a_2', \dots, a_c'\}\rangle = 0, \tag{24.2}$$

where c is the number of compatible properties.

Chapter 13 discussed the unitary transformations that represent tranlations and rotations. For an infinitesimal transformation,

$$\overline{X} = U^{-1}XU, \quad \overline{\langle\ |} = \langle\ |U, \quad \overline{|\ \rangle} = U^{-1}|\ \rangle, \tag{24.3}$$

where

$$U = 1 + \frac{i}{\hbar}G, \tag{24.4}$$

where $G = G^\dagger$ is the generator of the transformation. For a spatial translation of the origin through an amount $\delta\boldsymbol{\epsilon}$,

$$G_{\delta\boldsymbol{\epsilon}} = \delta\boldsymbol{\epsilon}\cdot\mathbf{P}, \tag{24.5}$$

where $\mathbf{P}$ is the linear momentum. For a rotation through an angle $\delta\boldsymbol{\omega}$, the generator is

$$G_{\delta\boldsymbol{\omega}} = \delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{24.6}$$

where $\mathbf{J}$ is the angular momentum. A scalar S does not change under a rotation, so

$$[S, \delta\boldsymbol{\omega}\cdot\mathbf{J}] = 0, \tag{24.7}$$

while a vector operator $\mathbf{V}$ transforms as

$$\frac{1}{i\hbar}[\mathbf{V}, \delta\boldsymbol{\omega}\cdot\mathbf{J}] = \delta\boldsymbol{\omega}\times\mathbf{V}. \tag{24.8}$$

In particular, $\mathbf{J}^2$ is a scalar,

$$[\mathbf{J}^2, \mathbf{J}] = 0, \tag{24.9}$$

while $\mathbf{J}$ is a vector, so

$$[J_x, J_y] = i\hbar J_z, \tag{24.10}$$

and so on, by cyclic permutations. Since $\mathbf{J}^2$ and J_z are compatible, we can specify states by values of both of these operators:

$$\mathbf{J}^2|jm\rangle = j(j+1)\hbar^2|jm\rangle, \quad J_z|jm\rangle = m\hbar|jm\rangle. \tag{24.11}$$

Because $J_+ = J_x + iJ_y$ and $J_- = J_x - iJ_y$ are raising and lowering operators,

$$J_+|jm\rangle = \hbar\sqrt{(j-m)(j+m+1)}|jm+1\rangle, \tag{24.12a}$$
$$J_-|jm\rangle = \hbar\sqrt{(j+m)(j-m+1)}|jm-1\rangle, \tag{24.12b}$$

where the factors are required so that the states are properly normalized, it is easy to see that j is a nonnegative integer, and m ranges by integer steps from $-j$ to j. For a given j, there are $2j+1$ values of m.

Chapter 14 approached the harmonic oscillator by considering the limit of large j. In this way we obtain raising and lowering operators

$$y|n\rangle = \sqrt{n}|n-1\rangle, \quad y^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{24.13}$$

These operators satisfy

$$[y, y^\dagger] = 1. \tag{24.14}$$

The states labelled by n are eigenvectors of $y^\dagger y$:

$$y^\dagger y|n\rangle = n|n\rangle. \tag{24.15}$$

Instead of the non-Hermitian operators y and $y^\dagger$ we can use Hermitian variables q and p,

$$y = \frac{q+ip}{\sqrt{2}}, \quad y^\dagger = \frac{q-ip}{\sqrt{2}}, \tag{24.16}$$

which satisfy

$$[q, p] = i. \tag{24.17}$$

We can introduce the Hamiltonian

$$\frac{q^2+p^2}{2} = y^\dagger y + \frac{1}{2}, \quad \left(\frac{q^2+p^2}{2}\right)' = n + \frac{1}{2}. \tag{24.18}$$

On a position eigenstate,

$$\langle q'|p = \frac{1}{i}\frac{\partial}{\partial q'}\langle q'|. \tag{24.19}$$

From the commutation relation we derive the uncertainty relation

$$\Delta q \Delta p \geq \frac{1}{2}. \tag{24.20}$$

The ground state is characterized by $y|0\rangle = 0$, from which we deduce the wave-function

$$\psi_0(q') = \langle q'|0\rangle = \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}q'^2}. \tag{24.21}$$

The general harmonic oscillator wavefunction is

$$\psi_n(q') = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} H_n(q') e^{-\frac{1}{2}q'^2}, \tag{24.22}$$

where $H_n(q')$ are the Hermite polynomials,

$$H_0(q') = 1, \quad H_1(q') = 2q', \quad H_2(q') = 4q'^2 - 2, \ldots. \tag{24.23}$$

The wavefunctions are orthonormal,

$$\langle n|n'\rangle = \int_{-\infty}^{\infty} dq' \psi_n(q')^* \psi_{n'}(q') = \delta_{nn'}. \tag{24.24}$$

These dimensionless variables can be rescaled to physical coordinates and momentum,

$$\hat{p} = \sqrt{m\hbar\omega} p, \quad \hat{q} = \sqrt{\frac{\hbar}{m\omega}} q, \tag{24.25}$$

corresponding to the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2 = H\hbar\omega, \tag{24.26}$$

which yields the energy eigenvalues

$$\hat{E}_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{24.27}$$

Chapter 15 begins by deriving the momentum-space wavefunctions for the harmonic oscillator,

$$\psi_n(p') = \langle p'|n\rangle = \frac{(-1)^n}{\sqrt{\pi^{1/2} 2^n n!}} H_n(p') e^{-\frac{1}{2}p'^2}. \tag{24.28}$$

Then we show how to construct angular momentum in general from two oscillators. Construct a two-component object

$$y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix}, \tag{24.29}$$

where

$$[y_+, y_+^\dagger] = [y_-, y_-^\dagger] = 1, \quad [y_-, y_+] = [y_+, y_-^\dagger] = 0, \tag{24.30}$$

Then a general angular momentum can be written as

$$\frac{1}{\hbar}\mathbf{J} = y^\dagger \frac{1}{2}\boldsymbol{\sigma} y, \tag{24.31}$$

in terms of the spin-1/2 Pauli spin matrices.

Chapter 16 returns to translations, and suggests in terms of a position operator $\mathbf{R}$ we have a decomposition of angular momentum into orbital and spin angular momentum,

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \quad \mathbf{L} = \mathbf{R} \times \mathbf{P}. \tag{24.32}$$

$\mathbf{R}$ and $\mathbf{P}$ satisfy

$$[R_k, P_l] = i\hbar\delta_{kl}, \tag{24.33}$$

while since the order of translations does not matter,

$$[P_k, P_l] = 0. \tag{24.34}$$

Analogously,

$$[R_k, R_l] = 0. \tag{24.35}$$

The spin $\mathbf{S}$ commutes with both $\mathbf{R}$ and $\mathbf{P}$, but by itself satisfies the commutation relations of angular momentum,

$$\mathbf{S} \times \mathbf{S} = i\hbar\mathbf{S}. \tag{24.36}$$

Chapter 17 extends the transformations to Galilean transformations. A boost is a transformation to a coordinate frame moving with velocity $\delta\mathbf{v}$ with respect to the original frame. The generator is given by

$$G_{\delta\mathbf{v}} = \delta\mathbf{v} \cdot \mathbf{N}. \tag{24.37}$$

$\mathbf{N}$ must be a vector, which supplies

$$[\mathbf{J}, \delta\mathbf{v} \cdot \mathbf{N}] = i\hbar\delta\mathbf{v} \times \mathbf{N}, \tag{24.38}$$

which is reminiscent of

$$[\mathbf{J}, \delta\boldsymbol{\epsilon} \cdot \mathbf{P}] = i\hbar\delta\boldsymbol{\epsilon} \times \mathbf{P}. \tag{24.39}$$

The boost generators satisfy the commutation relations

$$[N_k, N_l] = 0, \tag{24.40}$$

while the mass M of the system is introduced through

$$[P_k, N_l] = i\hbar M\delta_{kl}. \tag{24.41}$$

The latter reflects the phase ambiguity in defining generators. Since a boost is like a translation that grows with time, we expect

$$\delta_{\delta\mathbf{v}}\mathbf{R} = \delta\mathbf{v}t = \frac{1}{i\hbar}[\mathbf{R}, \delta\mathbf{v} \cdot \mathbf{N}]. \tag{24.42}$$

This leads to the construction

$$\mathbf{N} = \mathbf{P}t - M\mathbf{R}. \tag{24.43}$$

Time translations are given in terms of the Hamiltonian,

$$G_{\delta t} = -\delta t\, H, \tag{24.44}$$

For an isolated system, the Hamiltonian is invariant under spatial translations, or

$$[P_k, H] = 0. \tag{24.45}$$

If the system is also rotationally invariant,

$$[J_k, H] = 0, \tag{24.46}$$

that is, H is a scalar. Under a boost,

$$\frac{1}{i\hbar}[\mathbf{N}, H] + \mathbf{P} = 0. \tag{24.47}$$

In **Chapter 18**, we continue to study dynamics, and show for a dynamical variable,

$$\frac{d}{dt}v(t) = \frac{1}{i\hbar}[v(t), H]. \tag{24.48}$$

The general Heisenberg equation for a function of a dynamical variable is

$$\frac{d}{dt}F = \frac{\partial}{\partial t}F + \frac{1}{i\hbar}[F, H]. \tag{24.49}$$

From the boost commutator, we then learn that the velocity of the center of mass of the system is a constant, $\mathbf{P}/M$. From this we conclude that

$$H = \frac{\mathbf{P}^2}{2M} + H_{\text{int}}, \tag{24.50}$$

where H_{int} does not depend on either the center of mass coordinate $\mathbf{R}$ or the total momentum $\mathbf{P}$.

Chapter 19 deals with the hydrogen atom,

$$H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}, \tag{24.51}$$

where $\mathbf{p}$ is the relative momentum of the electron and nucleus, μ is the reduced mass, and Ze is the charge of the nucleus. This system is rotationally invariant, so the orbital angular momentum must be conserved,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \frac{d}{dt}\mathbf{L} = 0. \tag{24.52}$$

But there is another constant of the motion, the axial vector,

$$A = \frac{r}{r} - \frac{\mathbf{p} \times \mathbf{L}}{\mu Z e^2}, \quad \frac{d}{dt}\mathbf{A} = 0. \tag{24.53}$$

Quantum mechanically, $\mathbf{A}$ must be modified to make it Hermitian,

$$\mathbf{p} \times \mathbf{L} \to \mathbf{p} \times \mathbf{L} - i\hbar\mathbf{p}. \tag{24.54}$$

We then can show that

$$\mathbf{J}^{(\pm)} = \frac{1}{2}\left(\mathbf{L} \pm \sqrt{\frac{\mu Z^2 e^4}{-2H}}\mathbf{A}\right) \tag{24.55}$$

form two independent angular momenta, for example,

$$[J_x^{(\pm)}, J_y^{(\pm)}] = i\hbar J_z^{(\pm)}, \quad [J_x^{(\pm)}, J_y^{(\mp)}] = 0. \tag{24.56}$$

From this we conclude that the energy eigenvalues are

$$E_n = -\frac{\mu Z^2 e^4}{2n^2\hbar^2}, \quad n = 1, 2, 3, \dots. \tag{24.57}$$

Here $n = 2j+1$, where j is the angular momentum quantum number associated with either $\mathbf{J}^2$,

$$(\mathbf{J}^{(\pm)})^{2\prime} = j(j+1)\hbar^2. \tag{24.58}$$

Therefore there are n^2 states with the same value of energy E_n. These states are characterized by angular momentum quantum numbers $l = 0, 1, \dots, n-1$, where for a given l there are $2l+1$ possible values of the magnetic quantum number m. The ground-state wavefunction is easily worked out by noting the state vector is annihilated by both $\mathbf{L}$ and $\mathbf{A}$,

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}, \tag{24.59}$$

in terms of the Bohr radius $a_0 = \hbar^2/\mu e^2$.

Chapter 20 deals with the addition of angular momentum. That is, if we have two independent systems characterized by angular momentum generators $\mathbf{J}_1$ and $\mathbf{J}_2$, the whole system has angular momentum

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2. \tag{24.60}$$

This means, in particular, that magnetic quantum numbers add:

$$m = m_1 + m_2. \tag{24.61}$$

The possible values of the total angular momentum quantum number j range from $j = j_1 + j_2$ down to $|j_1 - j_2|$. This is verified by counting,

$$(2j_1 + 1)(2j_2 + 1) = \sum_{|j_1 - j_2|}^{j_1 + j_2} (2j + 1). \tag{24.62}$$

Rotation matrices were the subject of **Chapter 21**. We showed how a general rotation is characterized by three Euler angles, ϕ, θ, ψ. A general rotation operator is

$$U(\phi,\theta,\psi) = e^{i\psi J_z/\hbar} e^{i\theta J_y/\hbar} e^{i\phi J_z/\hbar}, \tag{24.63}$$

corresponding to first rotating about the z axis through the angle ϕ, then about the new y axis by the angle θ, and finally about the final z axis by the angle ψ. Recalling that a general angular momentum can be constructed two spin-1/2 systems, described by oscillator variables, we have

$$|jm\rangle = \frac{(y_+^\dagger)^{j+m}(y_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle, \tag{24.64}$$

where

$$|0\rangle = |n_+ = 0, n_- = 0\rangle = |j=0, m=0\rangle. \tag{24.65}$$

Using this the general rotation matrix can be determined.

In **Chapter 22** we consider the most interesting case, that of the spherical harmonics, defined by

$$\langle l0|U(\theta,\phi)|lm\rangle = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi). \tag{24.66}$$

These are read off by expanding the generating function

$$\frac{1}{2^l l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi) \frac{(y_+)^{j+m}(y_-)^{j-m}}{\sqrt{(j+m)!(j-m)!}} \tag{24.67}$$

in terms of the null vector

$$\mathbf{a} = (-y_+^2 + y_-^2, -iy_+^2 - iy_-^2, 2y_+y_-). \tag{24.68}$$

From the generating function, we can prove the orthonormality condition,

$$\int d\Omega\, Y_{lm}^*(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}, \quad d\Omega = \sin\theta\, d\theta\, d\phi. \tag{24.69}$$

From the generating function expression we can also see

$$\langle 0|\frac{1}{2^l l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi)\langle lm|, \tag{24.70}$$

which allows us to show that on spherical harmonics

$$\mathbf{L} = \mathbf{r}\times\frac{\hbar}{i}\boldsymbol{\nabla} \tag{24.71}$$

acts just like angular momentum. In particular

$$(L_x \pm iL_y)Y_{lm} = \hbar\sqrt{(l\mp m)(l\pm m+1)}\,Y_{l,m\pm 1}. \tag{24.72}$$

Finally, in **Chapter 23** we return to the hydrogen atom. We show that the general form of the wavefunction is

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi). \tag{24.73}$$

Then we considered perturbations due to constant magnetic and electric fields. For the former, the Zeeman effect, the energies are shifted depending on the value of the magnetic quantum number,

$$E_{nm} = E_n - \mu_B Bm, \quad \mu_B = \frac{e\hbar}{2\mu c}, \tag{24.74}$$

the Bohr magneton. For the electric field, the Stark effect, we find, up to a unitary transformation, that the effective Hamiltonian is

$$H = H_0 + \frac{3}{4}\frac{Ze^2}{H_0}\mathbf{A}\cdot e\mathcal{E}, \tag{24.75}$$

in terms of the constant of the motion, the axial vector. The Stark shift is expressed in terms of the constituent magnetic quantum numbers, $m^{(\pm)}$:

$$E_{n;m^{(+)},m^{(-)}} = E_n - \frac{3}{4}\frac{na_0}{Z}e\mathcal{E}(m^{(+)} - m^{(-)}). \tag{24.76}$$