Introduction to Quantum Mechanics II
Quiz 8

Name:

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The ground-state wavefunction of the hydrogen atom is characterized by the properites

$$\mathbf{L}\psi_{100}(r) = 0, \quad \mathbf{A}\psi_{100}(r) = 0.$$

Here

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Z e^2}(\mathbf{p} \times \mathbf{L} - i\hbar\mathbf{p}),$$

where on wavefunctions $\mathbf{p}$ is realized by the differential operator

$$\mathbf{p} = \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}} = \frac{\hbar}{i}\boldsymbol{\nabla}.$$

Show that the ground-state wavefunction in the form

$$\psi_{100} = Ce^{-Zr/a_0},$$

where C is a normalization constant, satifies these two equations provided the Bohr radius a_0 has a certain value given in terms of $\hbar$, μ, and e.

Solution: Since

$$\boldsymbol{\nabla} e^{-Zr/a_0} = -\hat{\mathbf{r}}\frac{Z}{a_0}e^{-Zr/a_0},$$

then, because $\mathbf{r} \times \mathbf{r} = 0$, $\mathbf{L}\psi_{100} = 0$, and

$$\mathbf{A}\psi_{100} = \left(\frac{\mathbf{r}}{r} + \frac{i\hbar}{\mu Z e^2}\frac{\hbar}{i}\boldsymbol{\nabla}\right)\psi_{100} = \hat{\mathbf{r}}\left(1 - \frac{\hbar^2}{\mu e^2 a_0}\right)\psi_{100} = 0,$$

if $a_0 = \hbar^2/\mu e^2$.