Introduction to Quantum Mechanics II
Quiz 2

Name:

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Starting from ($J_\pm = J_x \pm iJ_y$))

$$\frac{1}{\hbar} J_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |jm \pm 1\rangle,$$

specialize to the spin-1/2 case, where $\mathbf{J} = \hbar\boldsymbol{\sigma}/2$, and where we denote

$$|1/2, +1/2\rangle = |+\rangle, \quad |1/2, -1/2\rangle = |-\rangle$$

. Evaluate

$$\frac{1}{2}(\sigma_x + i\sigma_y)|+\rangle,$$
$$\frac{1}{2}(\sigma_x + i\sigma_y)|-\rangle,$$
$$\frac{1}{2}(\sigma_x - i\sigma_y)|+\rangle,$$
$$\frac{1}{2}(\sigma_x - i\sigma_y)|-\rangle.$$

Express the coefficients appearing here as matrices,

$$\langle \pm | \frac{1}{2}(\sigma_x + i\sigma_y)|\pm\rangle, \quad \langle \pm | \frac{1}{2}(\sigma_x - i\sigma_y)|\pm\rangle,$$

where, in each case, the two $\pm$'s are independent of each other. Add and subtract these, to get matrices for σ_x and σ_y.

Solution: Let $\sigma_\pm = \sigma_x \pm i\sigma_y$. Then, using the formula,

$$\begin{aligned} \frac{1}{2}\sigma_+|+\rangle &= 0, \quad \frac{1}{2}\sigma_+|-\rangle = \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{3}{2}-\frac{1}{2}\right)}|+\rangle = |+\rangle, \\ \frac{1}{2}\sigma_-|-\rangle &= 0, \quad \frac{1}{2}\sigma_-|+\rangle = \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{3}{2}-\frac{1}{2}\right)}|-\rangle = |-\rangle, \end{aligned}$$

Thus

$$\langle+|\frac{\sigma_+}{2}|+\rangle = 0, \quad \langle-|\frac{\sigma_+}{2}|+\rangle = 0,$$
$$\langle+|\frac{\sigma_+}{2}|-\rangle = 1, \quad \langle-|\frac{\sigma_+}{2}|+\rangle = 0,$$

or

$$\frac{\sigma_+}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Similarly

$$\langle+|\frac{\sigma_-}{2}|-\rangle = 0, \quad \langle-|\frac{\sigma_-}{2}|-\rangle = 0,$$
$$\langle-|\frac{\sigma_-}{2}|+\rangle = 1, \quad \langle+|\frac{\sigma_-}{2}|+\rangle = 0,$$

or

$$\frac{\sigma_+}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\sigma_x = \frac{1}{2}(\sigma_+ + \sigma_-) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{1}{2i}(\sigma_+ - \sigma_-) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$