Introduction to Quantum Mechanics II
Quiz 13

Name:

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Use the generating function for the spherical harmonics

$$\frac{1}{2^l l!}\left[(-\psi_+^2+\psi_-^2)\frac{x}{r}+(-i\psi_+^2-i\psi_-^2)\frac{y}{r}+(2\psi_+\psi_-)\frac{z}{r}\right]^l$$
$$=\sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\phi)\frac{\psi_+^{l+m}\psi_-^{l-m}}{\sqrt{(l+m)!(l-m)!}},$$

where ψ_+ and ψ_- are two arbitrary complex numbers, to read off Y_{00} and Y_{11}. Compute the same two spherical harmonics from the explicit form

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}}e^{im\phi}\sqrt{\frac{(l-m)!}{(l+m)!}}(-\sin\theta)^m\left(\frac{d}{d\cos\theta}\right)^{l+m}\frac{(\cos^2-1)^l}{2^l l!}.$$

Solution: Set $l=0$ and read off from generating function

$$Y_{00}=\frac{1}{\sqrt{4\pi}}.$$

For $l=1, m=1$, look at the coefficient of ψ_+^2:

$$\sqrt{\frac{4\pi}{3}}\frac{1}{\sqrt{2}}Y_{11}(\theta,\phi)=-\frac{1}{2}\frac{x+iy}{r}=-\frac{1}{2}\sin\theta e^{i\phi},$$

or

$$Y_{11}(\theta,\phi)=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}.$$

The same results are obtained from the explicit expression. In particular,

$$Y_{11}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}e^{i\phi}\sqrt{\frac{1}{2}}(-\sin\theta)\left(\frac{d}{d\cos\theta}\right)^2\frac{\cos^2\theta-1}{2}=-\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta.$$