Introduction to Quantum Mechanics II
Quiz 12

Name:

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Let

$$J_y(\theta) = e^{-iJ_x\theta/\hbar} J_y e^{iJ_x\theta/\hbar},$$

and

$$J_z(\theta) = e^{-iJ_x\theta/\hbar} J_z e^{iJ_x\theta/\hbar}.$$

By differentiating with respect to θ, and solving the resulting differential equations, compute $J_y(\theta)$ and $J_z(\theta)$ in terms of J_y and J_z. Why is this obvious geometrically? What happens if J_y, J_z are replaced by components of any vector operator V_y, V_z?

Solution:

$$\frac{d}{d\theta} J_y(\theta) = \frac{i}{\hbar} e^{-iJ_x\theta/\hbar} [J_y, J_x] e^{iJ_x\theta/\hbar} = J_z(\theta),$$

$$\frac{d}{d\theta} J_z(\theta) = \frac{i}{\hbar} e^{-iJ_x\theta/\hbar} [J_z, J_x] e^{iJ_x\theta/\hbar} = -J_y(\theta).$$

Imposing the boundary conditions at $\theta = 0$, $J_z(0) = J_z$, $J_y(0) = J_y$, we get

$$J_y(\theta) = J_y \cos\theta + J_z \sin\theta, \quad J_z(\theta) = J_z \cos\theta - J_y \sin\theta,$$

which is exactly how a vector behaves under a rotation about the x axis through an angle θ. This also holds for any vector operator, since all we used in the above was

$$[V_y, J_x] = -i\hbar V_z, \quad]V_z, J_x] = i\hbar V_y,$$

so

$$V_y(\theta) = V_y \cos\theta + V_z \sin\theta, \quad V_z(\theta) = V_z \cos\theta - V_y \sin\theta,$$