Second Examination
Physics 4803
Introduction to Quantum Mechanics II

November 19, 2012

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions. Remember this is a closed book, closed notes, exam. *Good luck!*

1. Consider two independent angular momentum 1 systems, that is, two systems which have angular momentum quantum number $j_1 = 1$ and $j_2 = 1$, respectively. The total angular momentum of the composite system is $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$.

   (a) How many states of the combined system are there? Compute this number in two ways: (1) by considering each angular momentum separately, and (2) by adding up the states described by the total angular momentum.

   (b) Consider the states of the combined system with total angular momentum 2. What is the state $|j = 2, m = 2\rangle$ in terms of the states $|j_1 = 1, m_1\rangle|j_2 = 1, m_2\rangle$.

   (c) Apply the lowering operator

   $$\frac{1}{\hbar}J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}|jm-1\rangle$$

   to this state to compute $|2, 1\rangle$ in terms of the individual angular momentum states.

   (d) Repeat the process to compute $|2, 0\rangle$ in terms of individual angular momentum states.

(e) Likewise, compute $|2,-1\rangle$.

(f) Finally compute $|2,-2\rangle$.

(g) Verify that each of the states computed above is represented by a unit vector.

2. A vector operator $\mathbf{V}$ satisfies

$$[V_x, J_y] = i\hbar V_z,$$

and so on, by cyclic permutations of the indices, in term of the generator of rotations, the angular momentum $\mathbf{J}$. Consider the unitary transformation

$$V_x(\theta) = e^{-i\theta J_z/\hbar} V_x e^{i\theta J_z/\hbar}, \quad V_y(\theta) = e^{-i\theta J_z/\hbar} V_y e^{i\theta J_z/\hbar}.$$

(a) What is the physical meaning of this transformation?

(b) What is $dV_x(\theta)/d\theta$ and $dV_y(\theta)/d\theta$?

(c) What is the value of $V_x(0)$ and $V_y(0)$?

(d) Solve the system of coupled differential equations in part 2b subject to the initial conditions in part 2c. [Hint: Convert the system to a second-order differential equation.]

(e) Does this make sense geometrically?

3. Recall that we solved the hydrogen atom problem by defining two independent angular momenta

$$\mathbf{J}^{(\pm)} = \frac{1}{2}\left(\mathbf{L} \pm \sqrt{\frac{\mu Z^2 e^4}{-2H}}\mathbf{A}\right),$$

in terms of the orbital angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

and the axial vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Z e^2}(\mathbf{p} \times \mathbf{L} - i\hbar \mathbf{p}).$$

The square of these two angular momenta is common:

$$(\mathbf{J}^{(\pm)})^2 = \frac{1}{4}\left(\frac{\mu Z^2 e^4}{-2H} - \hbar^2\right).$$

The eigenvalues of $(\mathbf{J}^{(\pm)})^2$ are given in terms of an angular momentum quantum number j. [Do not prove any of these statements.]

(a) In terms of the eigenvalues of $(\mathbf{J}^{(\pm)})^2$ determine the energy eigenvalues, that is, the eigenvalues of the Hamiltonian H.

(b) The ground state, or lowest energy state, of the atom corresponds to $j = 0$, or

$$\mathbf{J}^{(\pm)}|j = 0\rangle = 0.$$

. Show that this means that

$$\mathbf{L}|j = 0\rangle = \mathbf{A}|j = 0\rangle = 0.$$

Write the latter equation as a differential equation for $\psi_{j=0}(\mathbf{r})$.

(c) Solve that equation for the ground-state wavefunction $\psi_{j=0}(\mathbf{r})$.

(d) Determine the normalization constant for this wavefunction.

(e) How many energy eigenstates are there with $j = 1/2$?

(f) How many of these states have $L_z = 0$?

(g) In terms of the conventional labelling, the states of the hydrogen atom are labelled by the principal quantum number n, the orbital angular momentum quantum number l, and the magnetic quantum number m: $|nlm\rangle$. Compute $A_z|200\rangle$ in terms of $|210\rangle$, using the decomposition into eigenstates of $J_z^{(\pm)}$. [Note that $|210\rangle$ is symmetric in the two constituent "spins," while $|200\rangle$ is antisymmetric. Why?] Given that

$$\psi_{200}(\mathbf{r}) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{2a_0}\right)^{3/2}\left(1 - \frac{Zr}{2a_0}\right)e^{-Zr/2a_0},$$

in terms of the Bohr radius $a_0 = \hbar^2/\mu e^2$, compute $\psi_{210}(\mathbf{r})$.

(h) Check that the 210 wavefunction so computed is properly normalized.