# First Examination
# Physics 4803
# Introduction to Quantum Mechanics II

October 10, 2012

**Instructions:** Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions. Remember this is a closed book, closed notes, exam. *Good luck!*

1. Using the commutation relations for angular momentum,

$$[J_x, J_y] = i\hbar J_z,$$

etc., *directly* compute

$$[J_z, J^2], \quad J^2 = J_x^2 + J_y^2 + J_z^2.$$

What does this say about $J_z$ and $J^2$ being compatible?

2. The harmonic oscillator is governed by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2,$$

where the operators $q$ and $p$ satisfy

$$[q, p] = i\hbar.$$

Introduce new variables $\tilde{q}$ and $\tilde{p}$ by

$$q = \sqrt{\frac{\hbar}{m\omega}}\tilde{q}, \quad p = \sqrt{m\hbar\omega}\tilde{p}.$$

(a) What is the commutator

$$[\tilde{q}, \tilde{p}]?$$

(b) What is $H$ expressed in terms of $\tilde{q}$ and $\tilde{p}$?

(c) The lowest energy state, or ground state, denoted by $|0\rangle$, is annihilated by $y = (\tilde{q} + i\tilde{p})/\sqrt{2}$,

$$y|0\rangle = 0.$$

Use this fact to construct a differential equation for the ground-state wavefunction,

$$\psi_0(\tilde{q}') = \langle \tilde{q}'|0\rangle,$$

where $\langle \tilde{q}'|$ is an eigenvector of the operator $\tilde{q}$:

$$\langle \tilde{q}'|\tilde{q} = \tilde{q}'\langle \tilde{q}'|.$$

. [Hint: What is $\langle \tilde{q}'|\tilde{p}$?]

(d) Solve this differential equation, up to a multiplicative constant, for $\psi_0(\tilde{q}')$.

(e) Determine the multiplicative constant by imposing the normalization condition

$$\int_{-\infty}^{\infty} d\tilde{q}' |\psi_0(\tilde{q}')|^2 = 1.$$

(f) What is the energy of the ground state, $E_0$?

(g) Now re-express the ground-state wavefunction in terms of the eigenvalues of the original variable $q$. What is the normalization factor for $\psi_0(q')$?

(h) Verify that the ground state wavefunction satisfies the eigenvalue condition (time-independent Schrödinger equation),

$$\langle q'|H|0\rangle = E_0\langle q'|0\rangle,$$

by writing this as a differential equation for $\psi_0(q')$.

3. An isolated system is described by some Hamiltonian $H$. The statement that this is invariant under spatial translations is

$$\frac{1}{i\hbar}[\mathbf{P}, H] = 0, \tag{1}$$

the statement that it is invariant under rotations is

$$\frac{1}{i\hbar}[\mathbf{J}, H] = 0, \tag{2}$$

and the statement that it is invariant under Galilean transformations (boosts) is

$$\frac{1}{i\hbar}[\mathbf{N}, H] + \mathbf{P} = 0, \tag{3}$$

where $\mathbf{P}$ is the momentum operator, $\mathbf{J}$ is the angular momentum operator, and $\mathbf{N}$ is the boost generator.

(a) What does the first commutator (1) say about the dependence of $H$ on (center of mass) position $\mathbf{R}$?

(b) What does the second commutator (2) say about the dependence of $H$ on $\mathbf{P}$?

(c) If we write $\mathbf{N} = \mathbf{P}t - M\mathbf{R}$, from the third commutator (3) determine the dependence of $H$ on $\mathbf{P}$.