Introduction to Quantum Mechanics II
8th Homework Assignment
Due: Friday, December 7, 2012

December 3, 2012

1. Show that the wavefunctions given in class for $n = 1, 2$, in Eqs. (23.6) and (23.7), agree with those found earlier in lecture [(19.63) and (19.85)] and in Problems 3 and 4 of Assignment #5.

2. The Laplacian in spherical polar coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

The solid harmonics are related to the spherical harmonics by

$$\mathcal{Y}_{lm}(\mathbf{r}) = \left\{\begin{array}{c} r^l \\ r^{-l-1} \end{array}\right\} Y_{lm}(\theta,\phi).$$

The solid harmonics are solutions to Laplace's equation,

$$\nabla^2 \mathcal{Y}_{lm}(\mathbf{r}) = 0, \quad r \neq 0.$$

Show that

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\left\{\begin{array}{c} r^l \\ r^{-l-1} \end{array}\right\} = l(l+1)\left\{\begin{array}{c} r^l \\ r^{-l-1} \end{array}\right\}.$$

Then the differential equation satisfied by the spherical harmonics is

$$\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right] Y_{lm}(\theta,\phi) = 0.$$

3. The energy eigenvalue equation or the "time-dependent Schrödinger equation" for hydrogenic wavefunctions is

$$\left(-\frac{1}{2\mu}\nabla^2 - \frac{Ze^2}{r}\right)\psi_{nlm} = E_n\psi_{nlm}.$$

Using the result of Problem 2 verify that the wavefunctions for $n = 3$ given in class, Eqs. (23.8), satisfy this equation.

4. Suppose the Hamiltonian for a system depends on some parameter λ. The energy eigenvalue equation is

$$H(\lambda)|\lambda\rangle = E(\lambda)|\lambda\rangle.$$

Show that

$$\langle\lambda|\frac{\partial H(\lambda)}{\partial\lambda}|\lambda\rangle = \frac{\partial E(\lambda)}{\partial\lambda},$$

if λ is a normalized state vector.

5. Use the above Hellmann-Feynman theorem by regarding λ as Z in the hydrogenic atom, and thereby evaluate $\langle 1/r\rangle$ in an energy eigenstate.

6. Suppose we consider a small perturbation to a known Hamiltonian H_0,

$$H = H_0 + \lambda H_1.$$

Suppose the energies of the perturbed system can be developed in a power series in λ:

$$E = E_0 + \lambda E_2 + \lambda^2 E_2 + \ldots,$$

where E_0 is an energy eigenvalue of the original Hamiltonian,

$$H_0|0\rangle = E_0|0\rangle.$$

Then show from Problem 4 that

$$E_1 = \langle 0|H_1|0\rangle.$$

Apply this to the Zeeman effect, and rederive the result (23.14).

7. The Stark effect is more subtle. Show that the expectation value of the electric field perturbation (without loss of generality, assume $\mathcal{E}$ points in the z direction) $-e\mathcal{E} \cdot \mathbf{r}$ for the $2lm$ states is zero, but that in the superposition state

$$\frac{1}{\sqrt{2}}\left[|210\rangle + |200\rangle\right], \quad \text{or} \quad \psi = \frac{1}{\sqrt{2}}\left(\psi_{210} + \psi_{200}\right),$$

the expectation value of z is non zero, and thereby rederive the result (23.32) for $m^{(+)} = 1/2$ and $m^{(-)} = -1/2$.