Introduction to Quantum Mechanics II
6th Homework Assignment
Due: Friday, November 9, 2012

October 29, 2012

1. Consider adding two spin-1/2 systems,

$$\frac{1}{\hbar}\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}_1 + \frac{1}{2}\boldsymbol{\sigma}_2.$$

   Consider $\mathbf{J}^2$ and discuss the eigenvalues. Show that the eigenvalues satisfy

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)' = \begin{cases} 1, & j = 1, \\ -3, & j = 0, \end{cases}$$

   or

$$\left(\frac{1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{2}\right)' = \begin{cases} 1, & j = 1, \\ -1, & j = 0. \end{cases}$$

   This defines a operator

$$P = \frac{1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{2}.$$

   Check that this operator is Hermitian, and obeys $P^2 = 1$. Verify this last statement directly by proving

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)^2 = 3 - 2\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2.$$

2. Show that

$$\boldsymbol{\sigma}_1 P = P\boldsymbol{\sigma}_2, \quad \boldsymbol{\sigma}_2 P = P\boldsymbol{\sigma}_1.$$

Given that $|\sigma'\sigma''\rangle$ obeys

$$\sigma_{1z}|\sigma'\sigma''\rangle = \sigma'|\sigma'\sigma''\rangle, \quad \sigma_{2z}|\sigma'\sigma''\rangle = \sigma''|\sigma'\sigma''\rangle,$$

show that

$$\sigma_{2z}(P|\sigma'\sigma''\rangle) = \sigma'(P|\sigma'\sigma''\rangle), \quad \sigma_{1z}(P|\sigma'\sigma''\rangle) = \sigma''(P|\sigma'\sigma''\rangle).$$

The conclusion is that P is the $1 \leftrightarrow 2$ permutation operator. Is that consistent in the result in Problem 1?

3. Let $|a\rangle$, $|b\rangle$, $|c\rangle$, $|d\rangle$ be any four vectors for spin 1/2. Show that

$$\frac{1}{2}\left[\langle a|c\rangle\langle b|d\rangle + \langle a|\boldsymbol{\sigma}|c\rangle \cdot \langle b|\boldsymbol{\sigma}|d\rangle\right] = \langle a|d\rangle\langle b|c\rangle.$$

With $|d\rangle = |b\rangle$, $|c\rangle = |a\rangle$, all being unit vectors, derive

$$|\langle a|b\rangle|^2 = \frac{1}{2}\left[1 + \langle\boldsymbol{\sigma}\rangle_a \cdot \langle\boldsymbol{\sigma}\rangle_b\right].$$

Choose $|a\rangle$ to be a state in which the spin projection in the $\mathbf{n}_1$ direction is specified, and $|b\rangle$ to be a state in which the spin projection in the $\mathbf{n}_2$ direction is specified:

$$\langle a| = \langle(\boldsymbol{\sigma}\cdot\mathbf{n}_1)' = \sigma'|, \quad |b\rangle = |(\boldsymbol{\sigma}\cdot\mathbf{n}_2)' = \sigma''\rangle,$$

and obtain well-known probabilities.

4. With $|d\rangle = |a\rangle$ and $|c\rangle = |b\rangle$, both unit vectors, show

$$|\langle a|b\rangle|^2 = 2 - |\langle a|\boldsymbol{\sigma}|b\rangle|^2 = 2 - \langle a|\boldsymbol{\sigma}|b\rangle^* \cdot \langle a|\boldsymbol{\sigma}|b\rangle.$$

With the same choice of $\langle a|$ and $|b\rangle$ as in Problem 3, show for $\mathbf{n}_1$ and $\mathbf{n}_2$ in the x-y plane, show that, because σ_z anticommutes with σ_x and σ_y that

$$|\langle(\boldsymbol{\sigma}\cdot\mathbf{n}_1)'|(\boldsymbol{\sigma}\cdot\mathbf{n}_2)''\rangle|^2 + |\langle(\boldsymbol{\sigma}\cdot\mathbf{n}_1)'|\sigma_z|(\boldsymbol{\sigma}\cdot\mathbf{n}_2)''\rangle|^2 = 1.$$

Therefore it must be true that

$$|\langle(\boldsymbol{\sigma}\cdot\mathbf{n}_1)'|\sigma_x|(\boldsymbol{\sigma}\cdot\mathbf{n}_2)''\rangle|^2 + |\langle(\boldsymbol{\sigma}\cdot\mathbf{n}_1)'|\sigma_y|(\boldsymbol{\sigma}\cdot\mathbf{n}_2)''\rangle|^2 = 1.$$

Verify this by a similar argument.

5. Consider the 3 σ martices. Show that all the matrices $\sigma_y\boldsymbol{\sigma}$ are symmetrical. Express this as

$$\boldsymbol{\sigma}^T = -\sigma_y\boldsymbol{\sigma}\sigma_y,$$

where T means transpose.