Introduction to Quantum Mechanics II
5th Homework Assignment
Due: Friday, October 26, 2012

October 17, 2012

1. Show in general, based on the commutator $[x_k, p_l] = i\hbar\delta_{kl}$, that the square of the axial vector for the hydrogenic atom problem is
$$A^2 = 1 + \frac{2}{\mu Z^2 e^4} H(L^2 + \hbar^2),$$
where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and
$$H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}.$$
In class, or in the lecture notes, we only showed this was true for the kinetic energy part of the Hamiltonian. Show this result holds for the potential energy part as well.

2. The states of the hydrogenic atom are labelled by three quantum numbers, n, l, and m, the latter two referring to the angular momentum operator:
$$L_z|n,l,m\rangle = m\hbar|n,l,m\rangle \quad L^2|n,l,m\rangle = l(l+1)\hbar^2|n,l,m\rangle.$$
Since there are $2l+1$ values of m for a given value of l, determine the largest value of l for a given n so that the number of states belonging to a given value of n is n^2.

3. In class we determined that the wavefunction for the state with $n = 2$, $l = 1$, $m = 1$ is (up to a phase)
$$\psi_{211} = -\frac{1}{\sqrt{2\pi}}\left(\frac{Z}{2a_0}\right)^{3/2}\frac{Z}{2a_0}(x+iy)e^{-Zr/(2a_0)}.$$

Determine the two other states with $n = 2$, $l = 1$, that is, the wavefunctions $\psi_{2,1,0}$, $\psi_{2,1,-1}$, using the angular momentum lowering operator

$$L_- = L_x - iL_y.$$

This uses the fact that z-component of angular momentum, L_z, and the energy, H, are compatible physical properties, so that they may be specified simultaneously. Show that these states are orthogonal to each other,

$$\int (d\mathbf{r})\psi^*_{21m}(\mathbf{r})\psi_{21m'}(\mathbf{r}) = \delta_{mm'}.$$

4. In this problem we construct the 4th state of $n = 2$, namely that with $l = m = 0$. This can be determined from the $|2,1,0\rangle$ state determined in Problem 3. Show that

$$\frac{1}{\hbar}(J_z^{(-)} - J_z^{(+)})|210\rangle = |200\rangle,$$

up to a phase, since the operator does not change the value of m, while changing the symmetry of the two constituent "angular momenta." (Note for $n = 2$, $j = 1/2$.) Using the energy eigenvalue for $n = 2$, show that this means

$$|200\rangle = -2\hbar A_z|210\rangle.$$

Because $l = 0$, the wavefunction for this state must be independent of θ and ϕ, depending only on the radial coordinate r. It suffices, and simplifies the calculation, to evaluate the wavefunction at the pole,

$$x = y = 0, \quad z = r$$

. Then from

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Z e^2}(\mathbf{p} \times \mathbf{L} - i\hbar\mathbf{p}),$$

obtain ψ_{200} from ψ_{210} by applying a differential operator to the latter. Verify that that ψ_{200} is correctly normalized, and is orthogonal to ψ_{100}.