Introduction to Quantum Mechanics II
4th Homework Assignment
Due: WEDNESDAY, October 3, 2012

September 25, 2012

1. Recall that any Hermitian matrix can be brought into diagonal form by a suitable unitary transformation,
$$H_d = U^\dagger H U,$$
where H_d has zero for all elements not on the diagonal,
$$\langle n|H_d|m\rangle = \lambda_m \delta_{mn},$$
where $|m\rangle$ and λ_m are the eigenvectors and eigenvalues of H,
$$H|m\rangle = \lambda_m|m\rangle.$$
Prove that λ_m is real and if $\lambda_m \neq \lambda_n$, the corresponding eigevectors are orthogonal,
$$\langle m|n\rangle = 0, \quad \lambda_m \neq \lambda_n.$$

2. Show that for a Hermitian operator,
$$\mathrm{Tr}\, H = \sum_m \lambda_m.$$
What is $\det H$?

3. Show that for oscillator variables satisfying
$$[y, y^\dagger] = 1$$

that

$$e^{\lambda y^\dagger y} y e^{-\lambda y^\dagger y} = e^{-\lambda} y,$$

by constructing a differential equation. Evaluate

$$T(\lambda) = \mathrm{Tr}\left(e^{-\lambda y^\dagger y}\right),$$

by using differentiation and the property of the trace. To within a multiplicative constant, the result is

$$T(\lambda) = \frac{1}{1 - e^{-\lambda}}.$$

What does this say about the eigenvalues of $y^\dagger y$?

4. Show directly that a state with zero eigenvalue of $y^\dagger y$,

$$y^\dagger y|0\rangle = 0,$$

must obey

$$y|0\rangle = 0.$$

N.B. $y^\dagger$ does *not* have an inverse. If you add the fact that this state is unique, what further information follows from $T(\lambda)$?

5. In Einsteinian relativity the new feature is the finiteness of the speed of light c, and hence the abandonment of absolute simultaneity. Under a displacement, a rotation, or a boost the time coordinate changes by

$$\delta ct = \delta\epsilon^0 + \frac{1}{c}\delta\mathbf{v} \cdot r,$$

in addition to the change in the spatial coordinates,

$$\delta\mathbf{r} = \delta\boldsymbol{\epsilon} + \delta\boldsymbol{\omega} \times \mathbf{r} + \delta\mathbf{v}t.$$

It is convenient to designate the space-time coordinates collectively by

$$x^\mu = (ct, \mathbf{r}),$$

where $x^0 = -x_0 = ct$ and $x^k = x_k = \mathbf{r}_k$. The infinitesimal coordinate transformations of the Einsteinian relativity group are

$$\bar{x}^\nu = x^\nu - \delta x^\nu, \quad \delta x^\nu = \delta\epsilon^\nu + \delta\omega^{\mu\nu} x_\mu.$$

Here, repeated indices are summed over:

$$a^\mu b_\mu = -a^0 b^0 + \sum_{i=1}^{3} a_i b_j.$$

Here the 4 constant parameters $\delta\epsilon^\nu$ correspond to translations in space and time, while the 6 constant independent parameters in $\delta\omega^{\mu\nu}$, which are antisymmetrical,

$$\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu},$$

correspond to four-dimensional rotations. They correspond, in fact, to three-dimensional rotations, parameterized by $\delta\boldsymbol{\omega}$,

$$\delta\omega_{kl} = \epsilon_{klm}\delta\omega_m,$$

and to boosts, parameterized by $\delta\mathbf{v}$,

$$\delta\omega_{0k} = \frac{\delta v_k}{c}.$$

Show that the composition properties of the ten-parameter group of rotations, boosts, and displacements, corresponding to successive transformations 1, 2, 1^{-1}, 2^{-1}, are specified by

$$\begin{aligned} \delta_{[12]}\epsilon^\nu &= \delta_1\omega^{\mu\nu}\delta_2\epsilon_\mu - \delta_2\omega^{\mu\nu}\delta_1\epsilon_\mu, \\ \delta_{[12]}\omega^{\mu\nu} &= \delta_1\omega^{\mu\lambda}\delta_2\omega^\nu{}_\lambda - \delta_2\omega^{\mu\lambda}\delta_1\omega^\nu{}_\lambda. \end{aligned}$$

6. The generators of the unitary transformation induced by an infinitesimal coordinate transformation are comprised in

$$G = P^\mu\delta\epsilon_\mu + \frac{1}{2}J^{\mu\nu}\delta\omega_{\mu\nu} + \delta\phi 1.$$

The correspondence with the Galilean generators is

$$J_{kl} = \epsilon_{klm}J_m, \quad \frac{1}{c}J^{0k} = N_k, \quad cP^0 = H + Mc^2.$$

Show that, because no bilinear scalar can be formed from the vectors $\delta_{1,2}\epsilon^\mu$ and the tensors $\delta_{1,2}\omega^{\mu\nu}$ which is antisymmetrical, $\delta_{[12]}\phi = -\delta_{[21]}\phi$, we must conclude that

$$\delta_{[12]}\phi = 0.$$

Show from the results of Problem 5 that the full set of commutators for these generators is

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
\frac{1}{i\hbar}[P_\mu, J_{\kappa\lambda}] &= g_{\mu\lambda} P_\kappa - g_{\mu\kappa} P_\lambda, \\
\frac{1}{i\hbar}[J_{\mu\nu}, J_{\kappa\lambda}] &= g_{\mu\kappa} J_{\nu\lambda} - g_{\nu\kappa} J_{\mu\lambda} + g_{\nu\lambda} J_{\mu\kappa} - g_{\mu\lambda} J_{\nu\kappa},
\end{aligned}
$$

where $g_{\mu\nu}$ is the metric tensor specified by

$$g_{00} = -1, \quad g_{0k} = 0, \quad g_{ik} = \delta_{ik}.$$

Show that the commutators can also be written as

$$
\begin{aligned}
\frac{1}{i\hbar}[P^\nu, \frac{1}{2} J^{\kappa\lambda} \delta\omega_{\kappa\lambda}] &= \delta\omega^{\mu\nu} P_\mu, \\
\frac{1}{i\hbar}[J^{\mu\nu}, \frac{1}{2} J^{\kappa\lambda} \delta\omega_{\kappa\lambda}] &= \delta\omega^{\lambda\mu} J_\lambda{}^\nu + \delta\omega^{\lambda\nu} J^\mu{}_\lambda,
\end{aligned}
$$

indicating the response of vectors and tensors to infinitesimal Lorentz rotations (comprising three-dimensional rotations and boosts), and

$$
\begin{aligned}
\frac{1}{i\hbar}[P^\nu, P^\lambda \delta\epsilon_\lambda] &= 0, \\
\frac{1}{i\hbar}[J^{\mu\nu}, P^\lambda \delta\epsilon_\lambda] &= \delta\epsilon^\mu P^\nu - \delta\epsilon^\nu P^\mu,
\end{aligned}
$$

which gives the translational response of these operators. Show that when written in three-dimensional notation, all these commutators reproduce the Galilean ones, with two exceptions:

$$\frac{1}{i\hbar}[P_k, N_l] = \delta_{kl} \frac{P^0}{c}, \quad \frac{1}{i\hbar}[N_k, N_l] = -\frac{1}{c^2} J_{kl}.$$

Thus, show that in Galilean relativity, $\mathbf{J}/c^2$ is neglected, and H is neglected relative to Mc^2, giving the effective replacement of the operator P^0/c by the number M.