Introduction to Quantum Mechanics II
2nd Homework Assignment
Due: Friday, September 14, 2012

September 9, 2012

1. Given
$$\frac{1}{i}[q,p] = 1,$$
derive
$$e^{iq''p} q e^{-iq''p} = q + q'',$$
where q'' is a number, by differentiating with respect to q'', and derive
$$e^{-ip''q} p e^{ip''q} = p + p'',$$
where p'' is a number, by differentiating with respect to p''. What is
$$\left(\langle q'|e^{iq''p}\right) q,$$
and
$$\left(\langle p'|e^{-ip''q}\right) p.$$
Draw a conclusion from this, recalling that unitary transformations preserve lengths. From this, derive
$$\langle q'|p = \frac{1}{i}\frac{\partial}{\partial q'}\langle q'|,$$
$$\langle p'|q = i\frac{\partial}{\partial p'}\langle p'|.$$

2. Use the result of Problem 1 to evaluate

$$\frac{\partial}{\partial q'}\langle q'|p'\rangle, \quad \frac{\partial}{\partial p'}\langle q'|p'\rangle.$$

From these derive

$$\langle q'|p'\rangle = Ce^{iq'p'},$$

where C is a normalization constant. By looking at

$$\psi_0(q') = \frac{1}{\pi^{1/4}}e^{-q'^2/2}, \quad \psi_0(p') = \frac{1}{\pi^{1/4}}e^{-p'^2/2},$$

prove that $C = 1/\sqrt{2\pi}$.

3. Verify explicitly

$$\int_{-\infty}^{\infty} dq'\psi_0(q')^*\psi_2(q') = 0.$$

Compute $H_3(q')$, $H_4(q')$.

4. From

$$y|n\rangle = \sqrt{n}|n-1\rangle$$

derive

$$\frac{d}{dq'}H_n(q') = 2nH_{n-1}(q').$$

Check this for $n = 4, 3, 2, 1, 0$. From

$$y^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

derive

$$\left(2q' - \frac{d}{dq'}\right)H_n(q') = H_{n+1}(q').$$

Add the two statements to obtain

$$2q'H_n(q') = H_{n+1}(q') + 2nH_{n-1}(q').$$

This recursion relation gives a way of recursively calculating H_{n+1} in terms of H_n, H_{n-1}. Check this for $n = 3, 2, 1, 0$.

5. Use the results of Problem 4 to deduce the differential equation

$$\left(\frac{d^2}{dq'^2} - 2q'\frac{d}{dq'} + 2n\right) H_n(q') = 0.$$

Show the equivalence of this with

$$\left(\frac{d^2}{dq'^2} - q'^2 + 2n + 1\right) \psi_n(q') = 0.$$

This is the "time-independent Schrödinger equation" for the harmonic oscillator.