Introduction to Quantum Mechanics II
1st Homework Assignment
Due: Friday, August 31, 2012

August 20, 2012

1. Consider three numerical vectors, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0.$$

2. Now consider operators A, B, C. Show that

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

3. The multiplication property of the Pauli spin matrices can be written as

$$\boldsymbol{\sigma} \cdot \mathbf{a}\, \boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i \boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}).$$

From this, show that

$$\frac{1}{i}\left[\frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{a}, \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{b}\right] = \frac{1}{2}\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}).$$

More generally, what is

$$\frac{1}{i}[\mathbf{J} \cdot \mathbf{a}, \mathbf{J} \cdot \mathbf{b}]?$$

Use $A = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{a}$, $B = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{b}$, and $C = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{c}$ in the result of problem 2 to derive the result of problem 1.

4. Verify

$$[XY, Z] = X[Y, Z] + [X, Z]Y,$$
$$[X, YZ] = [X, Y]Z + Y[X, Z].$$

5. Vectors obey the transformation property

$$\frac{1}{i\hbar}[\mathbf{V}, \delta\boldsymbol{\omega} \cdot \mathbf{J}] = \delta\boldsymbol{\omega} \times \mathbf{V}.$$

If $\mathbf{V}_1$ and $\mathbf{V}_2$ are vectors, show that $\mathbf{V}_1 \times \mathbf{V}_2$ is a vector as well.

6. Consider successive infinitesimal coordinate rotations,

$$\mathbf{r}_1 = \mathbf{r} - \delta_1\boldsymbol{\omega} \times \mathbf{r}, \quad \mathbf{r}_{12} = \mathbf{r}_1 - \delta_2\boldsymbol{\omega} \times \mathbf{r}_1.$$

We can also consider doing the transformations in the opposite order,

$$\mathbf{r}_2 = \mathbf{r} - \delta_2\boldsymbol{\omega} \times \mathbf{r}, \quad \mathbf{r}_{21} = \mathbf{r}_2 - \delta_1\boldsymbol{\omega} \times \mathbf{r}_2.$$

To *first order* in the δ's, show that

$$\mathbf{r}_{12} = \mathbf{r}_{21} = \mathbf{r} - (\delta_1\boldsymbol{\omega} + \delta_2\boldsymbol{\omega}) \times \mathbf{r}.$$

To *second order* in the δ's, show that

$$\mathbf{r}_{12} - \mathbf{r}_{21} = -\delta_{[12]}\boldsymbol{\omega} \times \mathbf{r}, \quad \delta_{[12]}\boldsymbol{\omega} = \delta_1\boldsymbol{\omega} \times \delta_2\boldsymbol{\omega} = -\delta_{[21]}\boldsymbol{\omega}.$$

This is a very important statement about three-dimensional geometry. It shows that the order of geometrical operators is important. In mathematical terms, we say the group of rotations in *non-Abelian*.