Chapter 22

Spherical Harmonics

Let us now consider the above construction with $m = 0$ and $j = l$, an integer. Let us also restore the ϕ dependence, by undoing the substitution (21.81):

$$y_+ \to e^{i\phi/2} y_+, \quad y_- \to e^{-i\phi/2} y_-. \tag{22.1}$$

Then the operator structure in Eq. (21.83) becomes

$$\begin{aligned} &\frac{1}{l!}\left[\left(\cos\frac{\theta}{2} e^{i\frac{\phi}{2}} y_+ + \sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} y_-\right)\left(-\sin\frac{\theta}{2} e^{i\frac{\phi}{2}} y_+ + \cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} y_-\right)\right]^l \\ &= \sum_{m'=-l}^{l} \langle l0|U(\theta,\phi)|lm'\rangle \frac{y_+^{l+m'} y_-^{l-m'}}{\sqrt{(l+m')!(l-m')!}}. \end{aligned} \tag{22.2}$$

The product of the two factors in the square bracket is

$$-\frac{1}{2}\sin\theta e^{i\phi} y_+^2 + \frac{1}{2}\sin\theta e^{-i\phi} y_-^2 + \cos\theta\, y_+ y_-, \tag{22.3}$$

so the left side of Eq. (22.2) is

$$\frac{[-\sin\theta e^{i\phi} y_+^2 + \sin\theta e^{-i\phi} y_-^2 + 2\cos\theta y_+ y_-]^l}{2^l l!}. \tag{22.4}$$

Now go from polar to Cartesian coordinates:

$$\cos\theta = \frac{z}{r}, \quad \sin\theta e^{\pm i\phi} = \frac{x \pm iy}{r}, \tag{22.5}$$

so the quantity in the square brackets is $1/r$ times

$$-(x+iy)y_+^2 + (x-iy)y_-^2 + 2z y_+ y_- = \mathbf{r}\cdot\mathbf{a} \tag{22.6}$$

where the vector $\mathbf{a}$ has the components

$$a_x = -y_+^2 + y_-^2, \quad a_y = -iy_+^2 - iy_-^2, \quad a_z = 2y_+ y_-. \tag{22.7}$$

Note that

$$\mathbf{a}\cdot\mathbf{a} = (-y_+^2 + y_-)^2 + (-iy_+^2 - iy_-^2) + (2y_+y_-)^2 = 0, \tag{22.8}$$

so $\mathbf{a}$ is a vector of zero length, or a null vector. This is only possible because the components of $\mathbf{a}$ are complex. As a consequence, $(\mathbf{a}\cdot\mathbf{r})^l$ is a solution of Laplace's equation:

$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{r})^l = l(\mathbf{a}\cdot\mathbf{r})^{l-1}\mathbf{a}, \tag{22.9}$$

and then

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{r})^l = l(l-1)(\mathbf{a}\cdot\mathbf{r})^{l-2}\mathbf{a}\cdot\mathbf{a} = 0. \tag{22.10}$$

This leads to Legendre polynomials and spherical harmonics.

So we see that $(\mathbf{r}\cdot\mathbf{a})^l$, which is a special polynomial of degree l, is a solution of Laplace's equation. How many different, independent polynomials of degree l that satisfy Laplace's equation are there? Let's look at some examples:

$$l = 0: \quad \text{constant}, \quad 1 \text{ polynomial}, \tag{22.11a}$$

$$l = 1: \quad x, y, z, \quad 3 \text{ polynomials}, \tag{22.11b}$$

$$l = 2: \quad x^2, y^2, z^2, xy, xz, yz, \quad 6 \text{ polynomials}. \tag{22.11c}$$

For $l = 0, 1$, all the polynomials satisfy Laplace's equation. For $l = 2$, the xy, xz, and yz monomials obviously satisfy Laplace's equation, but x^2, y^2, and z^2 do not. If we consider a linear combination of these,

$$f = ax^2 + by^2 + cz^2, \quad \nabla^2 f = 2(a + b + c), \tag{22.12}$$

so the Laplacian on f vanishes only if $a + b + c = 0$. So there is one constraint, and there are 5 independent polynomials that satisfies Laplace's equation.

Let's do the count in general. The most general polynomial of degree l has the form

$$\sum_{m+n+p=l} A_{mnp} x^m y^n z^p. \tag{22.13}$$

How many different choices of non-negative integers, m, n, p, are there such that $m + n + p = l$? For a given p, how many different ways are there of satisfying $m + n = l - p$? Because m can take on all integer values from 0 to $l - p$, there are $l - p + 1$ ways. Then we sum over all possible values of p:

$$\sum_{p=0}^{l}(l - p + 1) = (l + 1)\frac{l + 2}{2}, \tag{22.14}$$

which is the number of terms in the sum times the average term. (This is just an arithmetic series.) This gives for $l = 0, 1, 2$ the number of polynomials being 1, 3, 6, as found above. Now, how many of these are solutions of Laplace's equation? When the Laplacian acts on a polynomial of degree l it produces a polynomial of degree $l - 2$. So we need to subtract the number of these, which is the number of conditions imposed in order to satisfy Laplace's equation. Therefore, the number of solutions of Laplace's equation which are polynomials of degree l is

$$\frac{(l+1)(l+2)}{2} - \frac{(l-1)l}{2} = 2l+1. \tag{22.15}$$

Back to the particular form of the solution of Laplace's equation $(\mathbf{r}\cdot\mathbf{a})^l$. How many of these are there? It is obvious from the structure that they are all there. But let us exhibit this explicitly. Return to the numerator of (22.4),

$$\begin{aligned}
&\left[-\sin\theta e^{i\phi}y_+^2 + \sin\theta e^{-i\phi}y_-^2 + 2\cos\theta y_+ y_-\right]^l \\
&= \left(-\frac{e^{-i\phi}y_-^2}{\sin\theta}\right)^l \left[\left(e^{i\phi}\sin\theta\frac{y_+}{y_-}\right)^2 - 2\cos\theta\sin\theta e^{i\phi}\frac{y_+}{y_-} - \sin^2\theta\right]^l \\
&= \left(-\frac{e^{-i\phi}y_-^2}{\sin\theta}\right)^l \left[\left(e^{i\phi}\sin\theta\frac{y_+}{y_-} - \cos\theta\right)^2 - 1\right]^l.
\end{aligned} \tag{22.16}$$

We expand in y_+/y_- in order to pick off $\langle l0|U(\theta,\phi)|lm\rangle$.

We may note that a Taylor series is another aspect of what we have been doing all along. Remember

$$e^{iq'p}qe^{-iq'p} = q + q', \tag{22.17}$$

which depends only on the algebraic relation between q and p,

$$\frac{1}{i}[q,p] = 1, \tag{22.18}$$

because

$$\frac{\partial}{\partial q'}\left[e^{iq'p}qe^{-iq'p}\right] = e^{iq'p}\frac{1}{i}[q,p]e^{-iq'p} = 1. \tag{22.19}$$

In general,

$$e^{iq'p}f(q)e^{-iq'p} = f(q+q'). \tag{22.20}$$

But now think of differential operators, as in $p = \frac{1}{i}\frac{\partial}{\partial q''}$, which is to say, generically,

$$e^{y\frac{\partial}{\partial x}}x\,e^{-y\frac{\partial}{\partial x}} = x + y, \tag{22.21}$$

where the exponentials should be thought of as represented by their power series,

$$e^{y\frac{\partial}{\partial x}} = \sum_{n=0}^{\infty}\frac{y^n}{n!}\left(\frac{\partial}{\partial x}\right)^n. \tag{22.22}$$

We can verify Eq. (22.21) by differentiating with respect to y:

$$\frac{\partial}{\partial y}\left[e^{y\frac{\partial}{\partial x}}xe^{-y\frac{\partial}{\partial x}}\right] = e^{y\frac{\partial}{\partial x}}\left[\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}\right]e^{-y\frac{\partial}{\partial x}} = 1, \tag{22.23}$$

which means that the coefficient of y is 1. Then

$$e^{y\frac{\partial}{\partial x}} f(x) e^{-y\frac{\partial}{\partial x}} = f\left(e^{y\frac{\partial}{\partial x}} x\, e^{-y\frac{\partial}{\partial x}}\right) = f(x+y), \tag{22.24}$$

which is really the same as the $q\ p$ statement (22.20). This can be rewritten as

$$f(x+y) = e^{y\frac{\partial}{\partial x}} f(x) = \sum_{n=0}^{\infty} \frac{y^n}{n!} \frac{d^n}{dx^n} f(x). \tag{22.25}$$

where the first equality is true because there is nothing for $\frac{\partial}{\partial x}$ to act on on the right, and the second comes from the series definition of the exponential (22.22). The Taylor series just gives the effect of displacement.

Now return to the expansion of Eq. (22.4) we wish to carry out:

$$\frac{1}{2^l l!}\left(-\frac{e^{-i\phi}}{\sin\theta} y_-^2\right)^l \left[\left(\cos\theta - e^{i\phi}\sin\theta \frac{y_+}{y_-}\right)^2 - 1\right]^l. \tag{22.26}$$

Think of this as the Taylor expansion (22.25) with $x = \cos\theta$, $y = -\sin\theta e^{i\phi} y_+/y_-$, and $n = l+m$, $m = -l, -l+1, \ldots$. Then the expansion of Eq. (22.26) is

$$\frac{1}{2^l l!}\left(-\frac{e^{-i\phi}}{\sin\theta} y_-^2\right)^l \sum_{m=-l}^{l} \frac{\left(-e^{i\phi}\sin\theta \frac{y_+}{y_-}\right)^{l+m}}{(l+m)!} \left(\frac{d}{d\cos\theta}\right)^{l+m} (\cos^2\theta - 1)^l. \tag{22.27}$$

Note that in the last factor the maximum power of of $\cos\theta$ is $2l$, which means that $m \leq l$. Now, we can read off the answer by comparing with

$$\sum_m \langle l0|U(\theta,\phi)|lm\rangle \frac{y_+^{l+m} y_-^{l-m}}{\sqrt{(l+m)!(l-m)!}}, \tag{22.28}$$

and then we obtain

$$\langle l0|U(\theta,\phi)|lm\rangle = e^{im\phi}\sqrt{\frac{(l-m)!}{(l+m)!}}(-\sin\theta)^m \left(\frac{d}{d\cos\theta}\right)^{l+m} \frac{(\cos^2\theta - 1)^l}{2^l l!}. \tag{22.29}$$

In homework, you derive the equivalent form

$$\langle l0|U(\theta,\phi)|lm\rangle = e^{im\phi}\sqrt{\frac{(l+m)!}{(l-m)!}}(\sin\theta)^{-m} \left(\frac{d}{d\cos\theta}\right)^{l-m} \frac{(\cos^2\theta - 1)^l}{2^l l!}, \tag{22.30}$$

which may be obtained by expanding in y_-/y_+ instead of y_+/y_-

Example: In the case $m = 0$,

$$\langle l0|U(\theta,\phi)|l0\rangle = \left(\frac{d}{d\cos\theta}\right)^l \frac{(\cos^2\theta - 1)^l}{2^l l!}. \tag{22.31}$$

These are Legendre's polynomials. For $l = 1$ this is

$$P_1(\cos\theta) = \cos\theta, \tag{22.32}$$

which is familiar as the probability amplitude $\langle 10, \bar{z}|10, z\rangle$. For $l = 2$ the Legendre polynomial is

$$P_2(\cos\theta) = \frac{d^2}{d\cos\theta^2}\left(\frac{\cos^4\theta - 2\cos^2\theta + 1}{8}\right) = \frac{1}{2}(3\cos^2\theta - 1). \tag{22.33}$$

In general, what we have been studying are polynomial solutions of Laplace's equation:

$$\nabla^2(\text{polynomial of degree } l) = 0, \tag{22.34}$$

which polynomial we write as a solid harmonic

$$r^l f(\theta, \phi), \tag{22.35}$$

where the function of θ and ϕ is a surface or spherical harmonic. The standard notation for spherical harmonics is

$$\langle l0|U(\theta,\phi)|lm\rangle = \sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\phi). \tag{22.36}$$

The spherical harmonics are normalized such that

$$\int d\Omega\, Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}, \tag{22.37}$$

where $d\Omega$ is the element of solid angle, or the surface element on the unit sphere. In spherical polar coordinates,

$$d\Omega = d\theta\,\sin\theta d\phi = d\cos\theta\, d\phi. \tag{22.38}$$

The statement (22.37) says that the spherical harmonics form an orthonormal set of functions on the unit sphere.

We prove this statement by going back to the beginning, Eq. (22.2):

$$\frac{1}{2^l l!}\left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\phi)\frac{y_+^{l+m}y_-^{l-m}}{\sqrt{(l+m)!(l-m)!}}. \tag{22.39}$$

We recall that the null vector $\mathbf{a}$ is quadratic in the ys, and that here y_+ and y_- just play the role of identifying powers; the operator properties are irrelevant. We change the notation to emphasize this:

$$y_+ \to \psi_+, \quad y_- \to \psi_-, \tag{22.40}$$

where $\psi_\pm$ are just complex numbers. Consequently, from Eq. (22.7),

$$a_x = -y_+^2 + y_-^2 \to -\psi_+^2 + \psi_-^2, \tag{22.41}$$

etc. Let us define

$$\frac{\psi_+^{l+m}\psi_-^{l-m}}{\sqrt{(l+m)!(l-m)!}} \equiv \psi_{lm}. \tag{22.42}$$

Then Eq. (22.39) becomes

$$\frac{\left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}\right)^l}{2^l l!} = \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi)\psi_{lm}. \tag{22.43}$$

The complex conjugate of this *numerical* structure is

$$\frac{\left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}^*\right)^l}{2^l l!} = \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi)^*\psi^*_{lm}, \tag{22.44}$$

so if we multiply these two expressions together, and integrate over all solid angles, we get

$$\int d\Omega \frac{\left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}^*\right)^l}{2^l l!}\frac{\left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}\right)^{l'}}{2^{l'} l'!} = \sum_{mm'} \psi^*_{lm}\sqrt{\frac{4\pi}{2l+1}}\left[\int d\Omega\, Y_{lm}(\theta,\phi)^* Y_{l'm'}(\theta,\phi)\right] \times\sqrt{\frac{4\pi}{2l'+1}}\psi_{l'm'}. \tag{22.45}$$

What we are asked to evaluate is

$$\int d\Omega \left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}^*\right)^l \left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}\right)^{l'} = f(\mathbf{a},\mathbf{a}^*), \tag{22.46}$$

which is a function having l' factors of $\mathbf{a}$ and l factors of $\mathbf{a}^*$. But f must be a scalar, since the left side is independent of the coordinate system. f can only depend on

$$\mathbf{a}\cdot\mathbf{a} = 0, \quad \mathbf{a}^*\cdot\mathbf{a}^* = 0, \quad \mathbf{a}\cdot\mathbf{a}^* \neq 0. \tag{22.47}$$

So this is zero unless the number of $\mathbf{a}$s equals the number of $\mathbf{a}^*$s, so we conclude that $l = l'$:

$$\int d\Omega \left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}^*\right)^l \left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}\right)^{l'} = \delta_{ll'} C(\mathbf{a}\cdot\mathbf{a}^*)^l. \tag{22.48}$$

How do we calculate the constant C? Take a particular example of a null vector:

$$\mathbf{a} = (1, i, 0), \quad \mathbf{a}^* = (1, -i, 0), \quad \mathbf{a}\cdot\mathbf{a} = \mathbf{a}^*\cdot\mathbf{a}^* = 0, \quad \mathbf{a}\cdot\mathbf{a}^* = 2. \tag{22.49}$$

Putting this into Eq. (22.48), we find

$$\int \sin\theta\, d\theta\, d\phi\, (\sin\theta e^{-i\phi})^l (\sin\theta e^{i\phi})^l = C2^l. \tag{22.50}$$

Letting $z = \cos\theta$, this is the same as

$$2\pi\int_{-1}^{1} dz\,(1-z^2)^l = C2^l. \tag{22.51}$$

The integral is symmetric about $z = 0$; define

$$I_l = \int_0^1 dz\,(1 - z^2)^l. \tag{22.52}$$

The first three of these integrals are evaluated to be

$$I_0 = 1, \quad I_1 = \frac{2}{3}, \quad I_2 = \frac{8}{15}. \tag{22.53}$$

In homework, you will prove in general

$$I_l = \frac{(2^l l!)^2}{(2l+1)!}. \tag{22.54}$$

Therefore, we conclude that

$$C = 4\pi 2^l \frac{(l!)^2}{(2l+1)!} \tag{22.55}$$

For $l = l'$ our formula (22.45) reads

$$\frac{4\pi}{(2l+1)!}\frac{(\mathbf{a}\cdot\mathbf{a}^*)^l}{2^l} = \sum_{mm'} \psi^*_{lm} \frac{4\pi}{2l+1}\left[\int d\Omega\, Y_{lm}(\theta,\phi)^* Y_{lm'}(\theta,\phi)\right]\psi_{lm'}, \tag{22.56}$$

where

$$\psi_{lm'} = \frac{\psi_+^{l+m'}\psi_-^{l-m'}}{\sqrt{(l+m')!(l-m')!}}, \quad \psi^*_{lm} = \frac{\psi_+^{*l+m}\psi_-^{*l-m}}{\sqrt{(l+m)!(l-m)!}}. \tag{22.57}$$

In the homework, you also prove

$$\mathbf{a}^*\cdot\mathbf{a} = 2(\psi^*\psi)^2, \tag{22.58}$$

which we here prove by brute force:

$$\begin{aligned}\mathbf{a}^*\cdot\mathbf{a} &= |-\psi_+^2 + \psi_-^2|^2 + |-i(\psi_+^2 + \psi_-^2)|^2 + |2\psi_+\psi_-|^2\\ &= 2[(\psi_+^*\psi_+)^2 + (\psi_-^*\psi_-)^2] + 4(\psi_+^*\psi_+)(\psi_-^*\psi_-)\\ &= 2[\psi_+^*\psi_+ + \psi_-^*\psi_-]^2,\end{aligned} \tag{22.59}$$

as stated. Thus we write Eq. (22.56) as

$$\frac{4\pi}{2l+1}\frac{(\psi_+^*\psi_+ + \psi_-^*\psi_-)^{2l}}{(2l)!} = \sum_{mm'} \psi^*_{lm} \frac{4\pi}{2l+1}\left[\int d\Omega\, Y_{lm}(\theta,\phi)^* Y_{lm'}(\theta,\phi)\right]\psi_{lm'}, \tag{22.60}$$

Expand this out by the binomial theorem—which is just Taylor's series:

$$(x+y)^n = \sum_{k=0} \frac{y^k}{k!}\left(\frac{d}{dx}\right)^k x^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k. \tag{22.61}$$

By substituting

$$n \to 2l, \quad k \to l-m, \quad n-k \to l+m, \tag{22.62}$$

we can now write Eq. (22.60) as

$$\begin{aligned}\frac{(\psi_+^*\psi_+ + \psi_-^*\psi_-)^{2l}}{(2l)!} &= \sum_m \frac{\psi_+^{*l+m}\psi_-^{*l-m}\psi_+^{l+m}\psi_-^{l-m}}{\sqrt{(l+m)!(l-m)!}\sqrt{(l+m)!(l-m)!}}\\ &= \sum_{m=-l}^{l}\psi_{lm}^*\psi_{lm}\\ &= \sum_{mm'}\psi_{lm}^*\left[\int d\Omega\, Y_{lm}(\theta,\phi)^* Y_{lm'}(\theta,\phi)\right]\psi_{lm'}.\end{aligned} \tag{22.63}$$

Thus, we conclude as promised,

$$\int d\Omega\, Y_{lm}^*(\theta,\phi) Y_{lm'}(\theta,\phi) = \delta_{mm'}. \tag{22.64}$$

The spherical harmonics are orthonormal functions. What do they mean? For each l, we have $m = l, l-1, \dots, -l$. What kind of angular momentum does Y_{lm} represent? Reinsert the operators:

$$\frac{\left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}\right)^l}{2^l l!} = \sum_m \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi)\frac{y_+^{l+m}y_-^{l-m}}{\sqrt{(l+m)!(l-m)!}}. \tag{22.65}$$

Let these operators act on a state of zero angular momentum, where from Eq. (21.64),

$$\langle 0|\frac{y_+^{l+m}y_-^{l-m}}{\sqrt{(l+m)!(l-m)!}} = \langle lm|. \tag{22.66}$$

Therefore,

$$\langle 0|\frac{\left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}\right)^l}{2^l l!} = \sum_m \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi)\langle lm|. \tag{22.67}$$

What happens when we rotate the coordinate system? An infinitesimal coordinate rotation is given in terms of

$$U = 1 + \frac{i}{\hbar}\delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{22.68}$$

so because $\langle 0|U = \langle 0|$,

$$\langle 0|\frac{\left(\frac{\mathbf{r}}{r}\cdot\mathbf{a}\right)^l}{2^l l!}U = \langle 0|\frac{\left(\frac{\mathbf{r}}{r}\cdot\overline{\mathbf{a}}\right)^l}{2^l l!}, \tag{22.69}$$

where

$$\overline{\mathbf{a}} = U^{-1}\mathbf{a}U = \mathbf{a} - \delta\boldsymbol{\omega}\times\mathbf{a}. \tag{22.70}$$

Thus,

$$\langle 0|\frac{\left(\frac{\mathbf{r}}{r}\cdot(\mathbf{a}-\delta\boldsymbol{\omega}\times\mathbf{a})\right)^l}{2^l l!} = \sum_m \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi)\langle lm|U. \tag{22.71}$$

Now we make a compensating rotation by replacing

$$\frac{\mathbf{r}}{r} \to \frac{\mathbf{r}}{r} - \delta\boldsymbol{\omega}\times\frac{\mathbf{r}}{r}. \tag{22.72}$$

Then the scalar product is unchanged, because we rotate $\mathbf{r}$ and $\mathbf{a}$ the same way. On the right side of Eq. (22.71) we have two changes:

$$\langle lm| \to \langle lm|\left(1+\frac{i}{\hbar}\delta\boldsymbol{\omega}\cdot\mathbf{J}\right), \tag{22.73a}$$

$$\mathbf{r} \to \mathbf{r}-\delta\boldsymbol{\omega}\times\mathbf{r}, \tag{22.73b}$$

the latter of which changes θ and ϕ. In terms of infinitesimal changes,

$$\delta\langle lm| = \langle lm|\frac{i}{\hbar}\delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{22.74a}$$

$$\delta Y_{lm}(\mathbf{r}) = (\delta\mathbf{r}\cdot\boldsymbol{\nabla})Y_{lm}(\mathbf{r}) = -(\delta\boldsymbol{\omega}\times\mathbf{r})\cdot\boldsymbol{\nabla}Y_{lm}(\mathbf{r}). \tag{22.74b}$$

Therefore, Eq. (22.67) is invariant under these two changes,

$$0 = \sum_m \sqrt{\frac{4\pi}{2l+1}}\left[-(\delta\boldsymbol{\omega}\times\mathbf{r})\cdot\boldsymbol{\nabla}Y_{lm}(\theta,\phi)\langle lm| + Y_{lm}(\theta,\phi)\langle lm|\frac{i}{\hbar}\delta\boldsymbol{\omega}\cdot\mathbf{J}\right], \tag{22.75}$$

which implies that

$$\sum_m(\mathbf{r}\times\frac{\hbar}{i}\boldsymbol{\nabla})Y_{lm}(\theta,\phi)\langle lm| = \sum_m Y_{lm}(\theta,\phi)\langle lm|\mathbf{J}. \tag{22.76}$$

Thus we conclude that

$$\mathbf{L} = \mathbf{r}\times\frac{\hbar}{i}\boldsymbol{\nabla} \tag{22.77}$$

represents on functions the action of $\mathbf{J}$ on states. This is analogous to the representation of linear momentum:

$$\frac{1}{i}\frac{\partial}{\partial q'}\langle q'| = \langle q'|p. \tag{22.78}$$

$\mathbf{L}$, a differential operator, represents, realizes, $\mathbf{J}$.

$\mathbf{L}$ has the same commutation relations as $\mathbf{J}$. We see this by computing

$$L_x L_y \sum_m Y_{lm}\langle lm| = L_x \sum_m Y_{lm}\langle lm|J_y = \sum_m Y_{lm}\langle lm|J_x J_y, \tag{22.79}$$

and similarly in the other order, so

$$[L_x, L_y] \sum_m Y_{lm} \langle lm| = \sum_m Y_{lm} \langle lm| [J_x, J_y] \\ = \sum_m Y_{lm} \langle lm| i\hbar J_z = i\hbar L_z \sum_m Y_{lm} \langle lm|, \quad (22.80)$$

so as a statement about differential operators,

$$[L_x, L_y] = i\hbar L_z, \quad \text{or} \quad \mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}. \quad (22.81)$$

Next,

$$L_z \sum_m Y_{lm} \langle lm| = \sum_m Y_{lm} \langle lm| J_z = \sum_m Y_{lm} \langle lm| m\hbar, \quad (22.82)$$

or

$$L_z Y_{lm} = m\hbar Y_{lm}. \quad (22.83)$$

That is, Y_{lm} is an eigenfunction of the differential operator L_z, with eigenvalue $m\hbar$. In the same way we prove

$$L^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}, \quad (22.84)$$

that is, Y_{lm} is an eigenfuction of the differential operator L^2, with eigenvalue $l(l+1)\hbar^2$,

Now recall the lowering operator,

$$\frac{1}{\hbar}(J_x - iJ_y)|lm\rangle = \sqrt{(l+m)(l-m+1)}|l, m-1\rangle, \quad (22.85)$$

or the adjoint statement,

$$\langle lm|(J_x + iJ_y)\frac{1}{\hbar} = \langle l, m-1|\sqrt{(l+m)(l-m+1)}. \quad (22.86)$$

Then

$$(L_x + iL_y) \sum_m Y_{lm} \langle lm| = \sum_m Y_{lm} \langle lm| (J_x + iJ_y), \quad (22.87)$$

so from Eq. (22.86)

$$(L_x + iL_y) \sum_m Y_{lm} \langle lm| = \sum_m Y_{lm} \hbar \langle l, m-1| \sqrt{(l+m)(l-m+1)} \\ = \hbar \sum_m Y_{l,m+1} \langle lm| \sqrt{(l-m)(l+m+1)}, \quad (22.88)$$

where in the last step we relabelled, or shifted $m \to m+1$. Thus, we conclude

$$(L_x + iL_y) Y_{lm} = \hbar \sqrt{(l-m)(l+m+1)} Y_{l,m+1}. \quad (22.89)$$

That is, acting to the right, $L_x + iL_y$ is just like $J_x + iJ_y$:

$$(J_x + iJ_y)|lm\rangle = \hbar \sqrt{(l-m)(l+m+1)} |l, m+1\rangle. \quad (22.90)$$

Similarly, we have the realization of the lowering operator to the right,

$$(L_x - iL_y)Y_{lm} = \hbar\sqrt{(l+m)(l-m+1)}Y_{l,m-1}, \tag{22.91}$$

analogous to Eq. (22.85). That is, $\mathbf{L} = \mathbf{r} \times \frac{\hbar}{i}\boldsymbol{\nabla}$, which indeed is a moment of momentum, is a *orbital* angular momentum, due to the motion of the particle.