Chapter 21

Rotation Matrices

We have talked at great length last semester and this about unitary transformations. The putting together of successive unitary transformations is of fundamental importance. We are now going to talk about a general rotation, described in terms of three Euler angles, as explained in Fig. 21.1 The three Euler angles (ϕ, θ, ψ) represent a general rotation of the coordinate system. First, the original coordinate system is rotated through an angle ϕ about the z axis; then, the new coordinate system is rotated through an angle θ about the new y' axis; finally, the 2nd coordinate system is rotated through an angle ψ about the second z'' axis.

We actually did this last semester for spin 1/2. Geometrically,

$$\sigma_{\bar{z}} = \sigma_z \cos\theta + \sigma_x \sin\theta \cos\phi + \sigma_y \sin\theta \sin\phi. \tag{21.1}$$

The last two terms are

$$\sin\theta\, \sigma_x(\cos\phi + i\sigma_z \sin\phi) = \sin\theta\, \sigma_x e^{i\sigma_z\phi} = e^{-i\sigma_z\phi/2} \sin\theta\, \sigma_x e^{i\sigma_z\phi/2}, \tag{21.2}$$

since σ_x anticommutes with σ_z. Then

$$\sigma_{\bar{z}} = e^{-i\sigma_z\phi/2}[\sigma_z \cos\theta + \sigma_x \sin\theta]e^{i\sigma_z\phi/2}, \tag{21.3}$$

where the quantity in brackets is

$$\sigma_z(\cos\theta + i\sigma_y \sin\theta) = \sigma_z e^{i\theta\sigma_y} = e^{-i\theta\sigma_y/2}\sigma_z e^{i\theta\sigma_y/2}. \tag{21.4}$$

In Eq. (21.3) we see the first transformation as a unitary transformation, that corresponds to a rotation about the z axis through an angle ϕ. In Eq. (21.4) we see the second transformation, the rotation about the y axis through an angle θ. We don't see the final rotation through the angle ψ here because σ_z commutes with itself. All together,

$$\sigma_{\bar{z}} = U^{-1}\sigma_z U, \quad U = e^{i\theta\sigma_y/2}e^{i\phi\sigma_z/2}. \tag{21.5}$$

This how we first learned about unitary transformations.

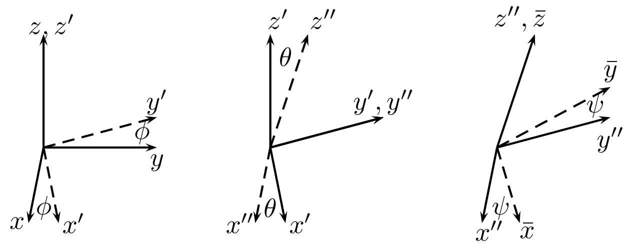

Figure 21.1: The left figure represents a rotation of the coordinate system about the z axis through an angle ϕ. The middle figure represents a subequent rotation about the y' axis through an angle θ. The figure on the right shows the final rotation, about the z'' axis through and angle ψ. The three rotations together leads to the final set of coordinate axes, designated by $\bar{x}$, $\bar{y}$, and $\bar{z}$

This result is very provocatively general. In general, is it true that we can replace

$$\frac{1}{2}\sigma_z \to \frac{1}{\hbar}J_z, \quad \frac{1}{2}\sigma_y \to \frac{1}{\hbar}J_y? \tag{21.6}$$

Let's check that this is true:

$$e^{-i\theta J_y/\hbar} J_z e^{i\theta J_y/\hbar} = J_z \cos\theta + J_x \sin\theta, \tag{21.7}$$

and also that

$$e^{-i\phi J_z/\hbar} J_x e^{i\phi J_z/\hbar} = J_x \cos\phi + J_y \sin\phi. \tag{21.8}$$

These are not independent statements, because the second can be obtained from the first by cyclic permuation,

$$z \to x, \quad x \to y, \quad y \to z. \tag{21.9}$$

So let's just verify Eq. (21.8). If this is true, it must follow from general commutation relations. Consider the derivative with respect to ϕ of the right-hand side of Eq. (21.8):

$$\frac{d}{d\phi}\left[e^{-i\phi J_z/\hbar}J_x e^{i\phi J_z/\hbar}\right] = e^{-i\phi J_z/\hbar}\frac{i}{\hbar}[J_x, J_z]e^{i\phi J_z/\hbar} = e^{-i\phi J_z/\hbar}J_y e^{i\phi J_z/\hbar}. \tag{21.10}$$

Let us denote the quantity differentiated by

$$e^{-i\phi J_z/\hbar}J_x e^{i\phi J_z/\hbar} = J_x(\phi). \tag{21.11}$$

The initial conditions are at $\phi = 0$:

$$J_x(0) = J_x, \quad \frac{dJ_x}{d\phi}(\phi = 0) = J_y, \tag{21.12}$$

so the solution to the differential equation (21.10) is

$$J_x(\phi) = e^{-i\phi J_z/\hbar}J_x e^{i\phi J_z/\hbar} = J_x \cos\phi + J_y \sin\phi. \tag{21.13}$$

This verified the desired relation. In exactly the same way

$$J_y(\phi) = e^{-i\phi J_z/\hbar}J_y e^{i\phi J_z/\hbar} = J_y \cos\phi - J_x \sin\phi. \tag{21.14}$$

These equations holds for any vector, since all we used was the commutation relation, for example,

$$[V_x, J_z] = -i\hbar V_y, \tag{21.15}$$

etc.

Think about the general rotation described in Fig. 21.1. Let U_1 be the unitary operator corresponding to the first rotation. U_1 changes states and operators according to

$$\overline{\langle\ |} = \langle\ |U_1, \quad \overline{X} = U_1^{-1}XU_1. \tag{21.16}$$

Here to describe the rotation about the z axis through the angle ϕ,

$$U_1 = e^{i\phi J_z/\hbar}. \tag{21.17}$$

The second rotation is about the new y' axis through an angle θ, where the y' axis was obtained from the y axis by the first rotation about the z axis. Thus

$$\overline{U}_2 = U_1^{-1}U_2U_1. \tag{21.18}$$

The transformation is described relative to the previous coordinate system. U_1 changes a rotation about the y axis to a rotation about the y' axis. The net result of these two rotations is

$$U_1\overline{U}_2 = U_1U_1^{-1}U_2U_1 = U_2U_1, \tag{21.19}$$

that is, the order of the rotations is reversed! Indeed, for spin 1/2 we just found

$$U = e^{i\theta\sigma_y/2}e^{i\phi\sigma_z/2}; \tag{21.20}$$

these transformations are all defined in the original coordinate system. For the sequence of three rotations described in Fig. 21.1, we have

$$U_1\overline{U}_2\overline{\overline{U}}_3, \tag{21.21}$$

where $\overline{U}_2$ describes a rotation about the y' axis, and $\overline{\overline{U}}_3$ represents a rotation about the z'' axis, which axis is produced by the two previous transformations. So

$$\overline{\overline{U}}_3 = (U_1\overline{U}_2)^{-1}U_3(U_1\overline{U}_2), \tag{21.22}$$

and the net transformation is

$$U_1\overline{U}_2\overline{\overline{U}}_3 = U_3(U_1\overline{U}_2) = U_3U_2U_1, \tag{21.23}$$

which is a general result: A sequence of transformations done in coordinate systems referring to previously transformed coordinate systems equals the transformation in the opposite order in the original coordinate system. So for a general rotation

$$U(\phi,\theta,\psi) = e^{i\psi J_z/\hbar}e^{i\theta J_y/\hbar}e^{i\phi J_z/\hbar}, \tag{21.24}$$

where each individual rotation operator refers to the original coordinate system. In fact, we showed this result last semester, in Assignment 5, problem 4, for spin 1/2:

$$U = e^{\frac{i}{2}\psi\sigma_z}e^{\frac{i}{2}\theta\sigma_y}e^{\frac{i}{2}\phi\sigma_z}. \tag{21.25}$$

What is the matrix of this? The matrix in the basis where σ_z is diagonal is composed of the elements

$$\langle\sigma'|U|\sigma''\rangle, \tag{21.26}$$

where the left vector corresponds to the state where $\sigma_z' = \sigma'$ and the right vector corresponds to the state where $\sigma_z' = \sigma''$. Immediately it is clear that

$$\langle\sigma'|U|\sigma''\rangle = e^{\frac{i}{2}\psi\sigma'}\langle\sigma'|e^{\frac{i}{2}\theta\sigma_y}|\sigma''\rangle e^{\frac{i}{2}\phi\sigma''}, \tag{21.27}$$

because the exponentials on the right and left simply record the values of σ_z on the right and left, respectively. Now

$$e^{\frac{i}{2}\theta\sigma_y} = \cos\frac{\theta}{2} + i\sigma_y\sin\frac{\theta}{2}, \tag{21.28}$$

where the matrix of σ_y is

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{21.29}$$

Thus the rotation matrix for spin 1/2 is

$$\langle\sigma'|U(\phi,\theta,\psi)|\sigma''\rangle = \begin{pmatrix} e^{i\psi/2}\cos\frac{\theta}{2}e^{i\phi/2} & e^{i\psi/2}\sin\frac{\theta}{2}e^{-i\phi/2} \\ -e^{-i\psi/2}\sin\frac{\theta}{2}e^{i\phi/2} & e^{-i\psi/2}\cos\frac{\theta}{2}e^{-i\phi/2} \end{pmatrix}. \tag{21.30}$$

This matrix gives probabilites. Recalling that $\overline{\langle\ |} = \langle\ |U$, we say that the state in which $\sigma'_z = \sigma'$ is transformed into the analogous state in which $\sigma'_{\bar z} = \sigma'$ (the same value) is

$$\langle\sigma'_z|U = \langle\sigma'_{\bar z}|, \tag{21.31}$$

or

$$\langle\sigma'_{\bar z}| = \langle\sigma'_z|U = \sum_{\sigma''_z}\langle\sigma'_z|U|\sigma''_z\rangle\langle\sigma''_z|, \tag{21.32}$$

where we see the appearance of the matrix elements we just computed. Explicitly,

$$\langle +,\bar z| = e^{\frac{i}{2}\psi}\left(\cos\frac{\theta}{2}e^{\frac{i}{2}\phi}\langle +,z| + \sin\frac{\theta}{2}e^{-\frac{i}{2}\phi}\langle -,z|\right), \tag{21.33a}$$

$$\langle -,\bar z| = e^{-\frac{i}{2}\psi}\left(-\sin\frac{\theta}{2}e^{\frac{i}{2}\phi}\langle +,z| + \cos\frac{\theta}{2}e^{-\frac{i}{2}\phi}\langle -,z|\right). \tag{21.33b}$$

In the first line here, we see just the probability amplitudes in Eq. (20.45). We want to do this for any angular momentum.

Remember how any angular momentum can be constructed in terms of spin 1/2, Eq. (15.59):

$$\frac{1}{\hbar}\mathbf{J} = y^\dagger\frac{1}{2}\boldsymbol{\sigma}y, \tag{21.34}$$

where y is a two-component operator,

$$y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix}. \tag{21.35}$$

Explicitly,

$$\frac{1}{\hbar}\mathbf{J} = \sum_{\sigma',\sigma''} y^\dagger_{\sigma'}\langle\sigma'|\frac{1}{2}\boldsymbol{\sigma}|\sigma''\rangle y_{\sigma''}. \tag{21.36}$$

Now, how do y, $y^\dagger$ respond to rotations of the coordinate system? Consider an infinitesimal rotation,

$$U = 1 + \frac{i}{\hbar}\delta\boldsymbol{\omega}\cdot\mathbf{J}. \tag{21.37}$$

Under a unitary transformation,

$$\bar X = U^{-1}XU, \tag{21.38}$$

amd if the transformation is infinitesimal, $U = 1 + iG/\hbar$,

$$\bar X = X - \delta X, \quad \delta X = \frac{1}{i\hbar}[X, G]. \tag{21.39}$$

Here

$$\delta y = \frac{1}{i\hbar}[y, \delta\boldsymbol{\omega}\cdot\mathbf{J}] = \frac{1}{i}[y, \sum_{\sigma',\sigma''} y^\dagger_{\sigma'}\langle\sigma'|\frac{1}{2}\boldsymbol{\sigma}\cdot\delta\boldsymbol{\omega}|\sigma''\rangle y_{\sigma''}]. \tag{21.40}$$

Remember that the matrix element appearing here is a number, not an operator. Now recall the operator properties (15.49a)–(15.49c) of the oscillator variables y, $y^\dagger$.

$$[y_{\sigma'}, y_{\sigma''}] = 0, \quad \text{or} \quad [y_+, y_-] = 0, \tag{21.41}$$

which says that y_+ and y_- are independent variables. On the other hand

$$[y_{\sigma'}, y^\dagger_{\sigma''}] = \delta_{\sigma',\sigma''}, \tag{21.42}$$

since

$$[y_+, y_+^\dagger] = 1, \quad [y_-, y_-^\dagger] = 1. \tag{21.43}$$

So

$$\delta y_{\sigma'} = \frac{1}{i}\sum_{\sigma''}\langle\sigma'|\frac{1}{2}\boldsymbol{\sigma}\cdot\delta\boldsymbol{\omega}|\sigma''\rangle y_{\sigma''}, \tag{21.44}$$

which says that the ys change into linear combinations of themselves. So if we write $\bar{y} = y - \delta y$,

$$\bar{y}_{\sigma'} = \sum_{\sigma''}\langle\sigma'|1 + \frac{i}{2}\boldsymbol{\sigma}\cdot\delta\boldsymbol{\omega}|\sigma''\rangle y_{\sigma''}, \tag{21.45}$$

where for spin 1/2

$$U = 1 + \frac{i}{2}\delta\boldsymbol{\omega}\cdot\boldsymbol{\sigma}; \tag{21.46}$$

this transformation is true for finite transformation,

$$\bar{y}_{\sigma'} = \sum_{\sigma''}\langle\sigma'|U|\sigma''\rangle y_{\sigma''}. \tag{21.47}$$

Here appears the known rotation matrix (21.30) for spim 1/2. This will tell us how an arbitary state transforms under a rotation.

The operators y transform the same as spin-1/2 states, as shown in Eqs. (21.33a) and (21.33b). [See Eq. (21.32).] Explicitly,

$$\bar{y}_+ = e^{\frac{i}{2}\psi}\left(\cos\frac{\theta}{2}e^{\frac{i}{2}\phi}y_+ + \sin\frac{\theta}{2}e^{-\frac{i}{2}\phi}y_-\right), \tag{21.48a}$$

$$\bar{y}_- = e^{-\frac{i}{2}\psi}\left(-\sin\frac{\theta}{2}e^{\frac{i}{2}\phi}y_+ + \cos\frac{\theta}{2}e^{-\frac{i}{2}\phi}y_-\right). \tag{21.48b}$$

The adjoint of this is

$$\bar{y}_+^\dagger = e^{-\frac{i}{2}\psi}\left(\cos\frac{\theta}{2}e^{-\frac{i}{2}\phi}y_+^\dagger + \sin\frac{\theta}{2}e^{\frac{i}{2}\phi}y_-^\dagger\right), \tag{21.49a}$$

$$\bar{y}_-^\dagger = e^{\frac{i}{2}\psi}\left(-\sin\frac{\theta}{2}e^{-\frac{i}{2}\phi}y_+^\dagger + \cos\frac{\theta}{2}e^{\frac{i}{2}\phi}y_-^\dagger\right). \tag{21.49b}$$

Under a unitary transformation, all algebraic properties are preserved. Now recall the commutation relations (21.41) and (21.42) for the ys and $y^\dagger$s, which also must be satisfied by the $\bar{y}$s and $\bar{y}^\dagger$s. Let's see this explicitly,

$$[\bar{y}_+, \bar{y}_+^\dagger] = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1, \tag{21.50a}$$

$$[\bar{y}_-, \bar{y}_-^\dagger] = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1, \tag{21.50b}$$

$$[\bar{y}_+, \bar{y}_-^\dagger] = e^{i\psi}\left(-\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 0. \tag{21.50c}$$

Remember, for a single oscillator variable, y, with $[y, y^\dagger] = 1$, we have [Eq. (15.1)]

$$|n\rangle = \frac{(y^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{21.51}$$

For angular momentum, we have two of these sets of ys, just as we saw in Eqs. (15.37a) and (15.37b):

$$|jm\rangle = |n_+, n_-\rangle, \quad n_+ = j + m, \quad n_- = j - m. \tag{21.52}$$

We are putting together n *spins of* 1/2,

$$n = n_+ + n_-, \tag{21.53}$$

where n_+ have spin up, n_- have spin down. The magnetic quantum number is

$$m = \frac{1}{2}(n_+ - n_-). \tag{21.54}$$

What is the largest m you can get? It is j, where

$$j = \frac{1}{2}n = \frac{1}{2}(n_+ + n_-). \tag{21.55}$$

This provides an interpretation of the $|n_+, n_-\rangle$ states in terms of two oscillators,

$$|jm\rangle = |n_+, n_-\rangle = \frac{(y_+^\dagger)^{n_+}}{\sqrt{n_+!}}\frac{(y_-^\dagger)^{n_-}}{\sqrt{n_+!}}|0, 0\rangle, \tag{21.56}$$

where the first operator creates n_+ up spins, the second n_- down spins. In terms of j and m,

$$|jm\rangle = \frac{(y_+^\dagger)^{j+m}(y_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle, \tag{21.57}$$

where

$$|0\rangle = |n_+ = 0, n_- = 0\rangle = |j = 0, m = 0\rangle. \tag{21.58}$$

Let's illustrate this for spin 1/2:

$$|1/2, +1/2\rangle = y_+^\dagger|0\rangle, \quad |1/2, -1/2\rangle = y_-^\dagger|0\rangle, \tag{21.59}$$

where the $y_+^\dagger$ operator creates one up spin, and $y_-^\dagger$ creates one down spin. Take the adjoint of this, in simplified notation:

$$\langle\sigma'_z| = \langle 0|y_{\sigma'}; \tag{21.60}$$

$y_{\sigma'}$ acting to the left creates a spin, up or down, depending on whether $\sigma' = \pm 1$.

Now it is obvious why $y_{\sigma'}$ transforms the same way as $\langle\sigma'_z|$:

$$\langle\sigma'_{\bar z}| = \langle\sigma'_z|U = \langle 0|y_{\sigma'}U = \langle 0|UU^{-1}y_{\sigma'}U = \langle 0|U\bar y_{\sigma'} = \langle 0|\bar y_{\sigma'}. \tag{21.61}$$

Here $\sigma'_{\bar z} = \sigma'_z = \sigma'$, since the two spin states have analogous properties in the rotated coordinate systems, related by Eulerian angles. Further, $\bar y_{\sigma'} = U^{-1}y_{\sigma'}U$, and

$$\langle 0|U = \langle 0|e^{i\psi J_z/\hbar}e^{i\theta J_y/\hbar}e^{i\phi J_z/\hbar} = \langle 0|, \tag{21.62}$$

because the "vacuum" state is the *unique* state in which we can specify J_x, J_y, J_z simultaneously,

$$\langle 0|J_z = 0, \quad \langle 0|(J_x \pm iJ_y) = \langle 0|J_\pm = \langle 0|\hbar y_\pm^\dagger y_\mp = 0 \Rightarrow \langle 0|J_x = \langle 0|J_y = 0, \tag{21.63}$$

which recalls the construction (15.48a) and (15.48b). So we conclude that the new spin-1/2 states are made from the new ys in the same way as the old spin-1/2 states are made from the old ys.

Now we want to do this in general, starting from

$$\langle jm| = \langle 0|\frac{(y_+)^{j+m}(y_-)^{j-m}}{\sqrt{(j+m)!(j-m)!}}. \tag{21.64}$$

What are these states in a different coordinate system?

$$\overline{\langle jm|} = \langle jm|U = \langle jm,\bar z|, \tag{21.65}$$

which relates the states referring to the z axis to those referring to the $\bar z$ axis. Because $\langle 0|U = \langle 0|$, we have

$$\langle 0|\frac{(y_+)^{j+m}(y_-)^{j-m}}{\sqrt{(j+m)!(j-m)!}}U = \langle 0|U^{-1}\frac{(y_+)^{j+m}(y_-)^{j-m}}{\sqrt{(j+m)!(j-m)!}}U, \tag{21.66}$$

so

$$\overline{\langle jm|} = \langle 0|\frac{(\bar y_+)^{j+m}(\bar y_-)^{j-m}}{\sqrt{(j+m)!(j-m)!}}. \tag{21.67}$$

Again, the new states are made from the new ys in just the same manner as the old states were made from the old ys. On the other hand,

$$\langle jm,\bar z| = \langle jm|U = \langle jm|U\sum_{j'm'}|j'm'\rangle\langle j'm'|, \tag{21.68}$$

where we have inserted a complete set of angular momentum states. But it is intuitively obvious that U does not change j:

$$\langle jm|U|j'm'\rangle = 0 \quad \text{if} \quad j' \neq j, \tag{21.69}$$

because this matrix element, apart from a phase, is

$$\langle jm|e^{i\theta J_y/\hbar}|j'm'\rangle, \tag{21.70}$$

and

$$[\mathbf{J}^2, \mathbf{J}] = 0, \tag{21.71}$$

so $\mathbf{J}^2$ commutes with any function of $\mathbf{J}$, in particular

$$[\mathbf{J}^2, U] = 0. \tag{21.72}$$

If we take the matrix element of this, it reads

$$0 = \langle jm|U(\mathbf{J})\mathbf{J}^2 - \mathbf{J}^2 U(\mathbf{J})|j'm'\rangle = \hbar^2[j'(j'+1) - j(j+1)]\langle jm|U(\mathbf{J})|j'm'\rangle, \tag{21.73}$$

which implies that the matrix element vanishes if $j' \neq j$. So we can write

$$\langle jm, \bar{z}| = \sum_{m'} \langle jm|U|jm'\rangle\langle jm'|, \tag{21.74}$$

which is the exact counterpart to what we had in Eq. (21.32) for spin 1/2:

$$\langle \sigma'_{\bar{z}}| = \sum_{\sigma''} \langle \sigma'|U|\sigma''\rangle\langle \sigma''_z|. \tag{21.75}$$

So our problem is to work out the $\langle jm|U|jm'\rangle$, a $(2j+1)\times(2j+1)$ matrix.

Write out explicitly, using Eqs. (21.48a) and (21.48b),

$$\begin{aligned}\langle jm, \bar{z}| &= \langle 0|\frac{(\bar{y}_+)^{j+m}(\bar{y}_-)^{j-m}}{\sqrt{(j+m)!(j-m)!}} \\ &= \langle 0|\frac{e^{im\psi}\left(\cos\frac{\theta}{2}e^{i\frac{\phi}{2}}y_+ + \sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}y_-\right)^{j+m}\left(-\sin\frac{\theta}{2}e^{i\frac{\phi}{2}}y_+ + \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}y_-\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}}.\end{aligned} \tag{21.76}$$

We have here all possible combinations of powers of y_+, y_-, with total power $2j$. To pick off the desired matrix element, we isolate the coefficient of

$$\langle jm'| = \langle 0|\frac{(y_+)^{j+m'}(y_-)^{j-m'}}{\sqrt{(j+m')!(j-m')!}}. \tag{21.77}$$

The first factor involving $y_\pm$ in Eq. (21.76) creates + spin along the $\bar{z}$ direction, the second factor creates − spin along the $\bar{z}$ direction. The desired matrix element is isolated as

$$\langle jm, \bar{z}| = \langle 0|\sum_{m'} \langle jm|U|jm'\rangle \frac{(y_+)^{j+m'}(y_-)^{j-m'}}{\sqrt{(j+m')!(j-m')!}}, \tag{21.78}$$

where the operator factor creates the original state $\langle jm'|$.

To determine the matrix element, we have an algebraic problem. First, we note that the dependence on ψ and ϕ is very simple,

$$\langle jm|e^{i\psi J_z/\hbar}e^{i\theta J_y/\hbar}e^{i\phi j_y/\hbar}|jm\rangle = e^{im\psi}\langle jm|e^{i\theta J_y/\hbar}|jm'\rangle e^{im'\phi}, \qquad (21.79)$$

so in Eq. (21.78) we see $e^{im\psi}$ as an overall factor, while $e^{i\phi/2}$ is associated with y_+, $e^{-i\phi/2}$ is associated with y_-; in fact,

$$e^{im'\phi}(y_+)^{j+m'}(y_-)^{j-m'} = \left(e^{i\phi/2}y_+\right)^{j+m}\left(e^{-i\phi/2}y_-\right)^{j-m}. \qquad (21.80)$$

So, we can redefine

$$e^{i\frac{\phi}{2}}y_+ \to y_+, \quad e^{-i\frac{\phi}{2}}y_- \to y_-, \qquad (21.81)$$

which is just another rotation about the z axis. Now we have, with

$$U(\theta) = e^{i\theta J_y/\hbar}, \qquad (21.82)$$

the operator appearing in Eq. (21.76) becoming

$$\begin{aligned}&\frac{(\cos\frac{\theta}{2}y_+ + \sin\frac{\theta}{2}y_-)^{j+m}(-\sin\frac{\theta}{2}y_+ + \cos\frac{\theta}{2}y_-)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\\ &= \sum_{m'=-j}^{j}\langle jm|U(\theta)|jm'\rangle\frac{y_+^{j+m'}y_-^{j-m'}}{\sqrt{(j+m')!(j-m')!}}.\end{aligned} \qquad (21.83)$$

The fact that the $y_\pm$ are operators is irrelevant to picking off the matrix element.

Let's do an example, for $j = 1$. For $m = 1$, the operator appearing on the left here is

$$\frac{(\cos\frac{\theta}{2}y_+ + \sin\frac{\theta}{2}y_-)^2}{\sqrt{2}} = \left(\frac{y_+^2}{\sqrt{2}}\right)\cos^2\frac{\theta}{2} + (y_+y_-)\frac{1}{\sqrt{2}}\sin\theta + \left(\frac{y_-^2}{\sqrt{2}}\right)\sin^2\frac{\theta}{2}, \qquad (21.84)$$

where the operators in parentheses on the right-side of this equality are those required to create the $m = 1, 0, -1$ state in the original coordinate system. For $m = 0$:

$$\begin{aligned}&\left(\cos\frac{\theta}{2}y_+ + \sin\frac{\theta}{2}y_-\right)\left(-\sin\frac{\theta}{2}y_+ + \cos\frac{\theta}{2}y_-\right)\\ &= -\left(\frac{y_+^2}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\sin\theta + (y_+y_-)\cos\theta + \left(\frac{y_-^2}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\sin\theta,\end{aligned} \qquad (21.85)$$

and, finally, for $m = -1$,

$$\frac{(-\sin\frac{\theta}{2}y_+ + \cos\frac{\theta}{2}y_-)^2}{\sqrt{2}} = \left(\frac{y_+^2}{\sqrt{2}}\right)\sin^2\frac{\theta}{2} - (y_+y_-)\frac{1}{\sqrt{2}}\sin\theta + \left(\frac{y_-^2}{\sqrt{2}}\right)\cos^2\frac{\theta}{2}. \qquad (21.86)$$

Thus we have compute all the matrix elements for $j = 1$:

$$\langle 1m|U(\theta)|1m'\rangle = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{\sqrt{2}}\sin\theta & \sin^2\frac{\theta}{2} \\ -\frac{1}{\sqrt{2}}\sin\theta & \cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ \sin^2\frac{\theta}{2} & -\frac{1}{\sqrt{2}}\sin\theta & \cos^2\frac{\theta}{2} \end{pmatrix}. \tag{21.87}$$

This should be familiar, since

$$\langle jm|U(\theta)|jm'\rangle = \langle jm, \bar{z}|jm', z\rangle, \tag{21.88}$$

which is just the transformation function between eigenstates in the two coordinate systems. The squares of these matrix elements are the well-known probabilities, given for example in Problem 2, Assignment 3, last semester. We are learning the algebraic signs that guarantee that the U matrix is *unitary.*