Chapter 20

Addition of Angular Momenta

Let's turn directly to the question of how we combine angular momentum. Consider two atoms, labelled 1 and 2, which are separated, so we can talk about measurements of their properties separately. Under a rotation, how does $\mathbf{J}_1$ change? By

$$\delta\mathbf{J}_1 = \frac{1}{i\hbar}[\mathbf{J}_1, \delta\boldsymbol{\omega}\cdot\mathbf{J}_1] = \delta\boldsymbol{\omega}\times\mathbf{J}_1, \tag{20.1}$$

which implies

$$\mathbf{J}_1\times\mathbf{J}_1 = i\hbar\mathbf{J}_1. \tag{20.2}$$

Atom 2 is outside the framework of this measurement. For example, the rotation of the coordinate system might be achieved physically by rotating magnets, which will not influence the isolated atom 2, so

$$\delta\mathbf{J}_2 = \frac{1}{i\hbar}[\mathbf{J}_2, \delta\boldsymbol{\omega}\cdot\mathbf{J}_1] = 0. \tag{20.3}$$

Thus, what we mean by independent augular momenta is

$$[J_{1k}, J_{2l}] = 0. \tag{20.4}$$

The angular momentum of these two atoms do not influence each other.

Similarly, we can measure atom 2 and let 1 be outside the realm of the measurement. So a rotation of the coordinate system describing atom 2 is described by

$$\delta\mathbf{J}_2 = \frac{1}{i\hbar}[\mathbf{J}_2, \delta\boldsymbol{\omega}\cdot\mathbf{J}_2] = \delta\boldsymbol{\omega}\times\mathbf{J}_2, \tag{20.5}$$

which implies

$$\mathbf{J}_2\times\mathbf{J}_2 = i\hbar\mathbf{J}_2. \tag{20.6}$$

Now rotation of the coordinate system describing atom 2 has no effect on 1:

$$\delta\mathbf{J}_1 = \frac{1}{i\hbar}[\mathbf{J}_1, \delta\boldsymbol{\omega}\cdot\mathbf{J}_2] = 0, \tag{20.7}$$

which again implies Eq. (20.4).

Now we think of 1 and 2 as being the whole system, described by a common coordinate system. Physically this could be realized by putting both atoms in the same magnetic field. Now the infinitesimal generator of rotations is

$$U = 1 + \frac{i}{\hbar} G. \tag{20.8}$$

For system 1 only, $G_1 = \delta\boldsymbol{\omega} \cdot \mathbf{J}_1$, and for system 2 only $G_2 = \delta\boldsymbol{\omega} \cdot \mathbf{J}_2$. We put both systems together by multiplying the unitary transformations, or adding the generators:

$$U = U_1 U_2 = 1 + \frac{i}{\hbar}(G_1 + G_2). \tag{20.9}$$

The generator of the common rotation of both atoms is

$$G = G_1 + G_2 = \delta\boldsymbol{\omega} \cdot (\mathbf{J}_1 + \mathbf{J}_2). \tag{20.10}$$

The angular momentum of the system as a whole is the sum of the angular momenta,

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2. \tag{20.11}$$

The change in the angular momenta induced by the rotation is

$$\delta\mathbf{J}_1 = \frac{1}{i\hbar}[\mathbf{J}_1, \delta\boldsymbol{\omega} \cdot \mathbf{J}] = \delta\boldsymbol{\omega} \times \mathbf{J}_1, \tag{20.12a}$$

$$\delta\mathbf{J}_2 = \frac{1}{i\hbar}[\mathbf{J}_2, \delta\boldsymbol{\omega} \cdot \mathbf{J}] = \delta\boldsymbol{\omega} \times \mathbf{J}_2, \tag{20.12b}$$

where we recognize that $\mathbf{J}_1$ rotates $\mathbf{J}_1$ and $\mathbf{J}_2$ rotates $\mathbf{J}_2$. Altogether then,

$$\delta\mathbf{J} = \frac{1}{i\hbar}[\mathbf{J}, \delta\boldsymbol{\omega} \cdot \mathbf{J}] = \delta\boldsymbol{\omega} \times \mathbf{J}, \tag{20.13}$$

or

$$\mathbf{J} \times \mathbf{J} = i\hbar\mathbf{J}, \tag{20.14}$$

where nothing but the total angular momentum appears.

Let the two systems both be angular momentum 1/2,

$$\frac{1}{\hbar}\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}_1 + \frac{1}{2}\boldsymbol{\sigma}_2; \tag{20.15}$$

this could be the spin ot the electron plus the spin of the proton in the hydrogen atom. These are independent:

$$[\sigma_{1k}, \sigma_{2l}] = 0. \tag{20.16}$$

Individually,

$$\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_1 = 2i\boldsymbol{\sigma}_1, \quad \boldsymbol{\sigma}_2 \times \boldsymbol{\sigma}_2 = 2i\boldsymbol{\sigma}_2. \tag{20.17}$$

In particular,

$$\frac{1}{\hbar}J_z = \frac{1}{2}\sigma_{1z} + \frac{1}{2}\sigma_{2z}, \tag{20.18}$$

so the eigenvalues satisfy

$$\frac{1}{\hbar}(J_z)' = m_1 + m_2 = \begin{cases} 1 & (1/2, 1/2) \\ 0 & (1/2, -1/2) \text{ or } (-1/2, 1/2) \\ -1 & (-1/2, -1/2) \end{cases} \tag{20.19}$$

since there are four possibilities of (m_1, m_2). In general, the magnetic quantum numbers add:

$$m = m_1 + m_2. \tag{20.20}$$

What about the total angular momentum? Recall the possible values of m are

$$m = j, j-1, j-2, \dots, -j, \tag{20.21}$$

$2j+1$ possibilities in all. Thus we are forced to conclude here that the $(m_1 = 1/2, m_2 = 1/2)$ state must also be the $j = 1$ $m = 1$ state, which we write in terms of state vectors as

$$|j=1, m=1\rangle = |m_1 = 1/2, m_2 = 1/2\rangle = |+\frac{1}{2}\rangle_1 |+\frac{1}{2}\rangle_2 = |+\frac{1}{2}\rangle_2 |+\frac{1}{2}\rangle_1. \tag{20.22}$$

This means, we put system 2 in state $|+1/2\rangle_2$ and system 1 in state $|+1/2\rangle_1$. We can write the product of vectors in either order since the systems are independent.

Now recall the machinery of angular momenta, in particular, the lowering operator,

$$\frac{1}{\hbar}J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}|jm-1\rangle, \tag{20.23}$$

or, in this case,

$$\frac{1}{\hbar}J_-|j=1, m=1\rangle = \sqrt{2}|j=1, m=0\rangle. \tag{20.24}$$

But on the other hand

$$\frac{1}{\hbar}J_- = \frac{1}{2}(\sigma_x - i\sigma_y)_1 + \frac{1}{2}(\sigma_x - i\sigma_y)_2, \tag{20.25}$$

where the spin operators for system 1 act only on the state 1, and those for 2 act only on state 2. In either case, from Eq. (20.23),

$$\frac{1}{2}(\sigma_x - i\sigma_y)|+1/2\rangle = |-1/2\rangle, \tag{20.26}$$

so we conclude

$$\sqrt{2}|j=1, m=0\rangle = |-1/2\rangle_1 |1/2\rangle_2 + |1/2\rangle_1 |-1/2\rangle_2 = |-\frac{1}{2}, \frac{1}{2}\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle, \tag{20.27}$$

where in the last we adopted the convention that the first spin projection refers to system 1, the second to system 2.

As we saw in Eq. (20.19) there are two ways of making $m = 0$, but a unique combination corresponds to the $j = 1$, $m = 0$ state,

$$|j = 1, m = 0\rangle = \frac{1}{\sqrt{2}} \left(| -\frac{1}{2}, +\frac{1}{2}\rangle + | +\frac{1}{2}, -\frac{1}{2}\rangle \right). \tag{20.28}$$

Now since there is only one way to make $m = -1$ we would anticipate

$$|j = 1, m = -1\rangle = | -\frac{1}{2}, -\frac{1}{2}\rangle, \tag{20.29}$$

but we can verify this directly, by applying the lowering operator again. On the one hand,

$$J_-|j = 1, m = 0\rangle = \sqrt{2}|j = 1, m = -1\rangle, \tag{20.30}$$

but on the other hand

$$\begin{aligned} &\left[\frac{1}{2}(\sigma_x - i\sigma_y)_1 + \frac{1}{2}(\sigma_x - i\sigma_y)_2\right] \frac{1}{\sqrt{2}} \left(| -\frac{1}{2}, \frac{1}{2}\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle \right) \\ &= \frac{1}{\sqrt{2}} \left(| -\frac{1}{2}, -\frac{1}{2}\rangle + | -\frac{1}{2}, -\frac{1}{2}\rangle \right) = \sqrt{2}| -\frac{1}{2}, -\frac{1}{2}\rangle, \end{aligned} \tag{20.31}$$

because, for example,

$$(\sigma_x - i\sigma_y)_1| -\frac{1}{2}, \frac{1}{2}\rangle = 0, \quad \frac{1}{2}(\sigma_x - i\sigma_y)_1|\frac{1}{2}, -\frac{1}{2}\rangle = | -\frac{1}{2}, -\frac{1}{2}\rangle. \tag{20.32}$$

Thus Eq. (20.29) is verified.

There is one more state, the remaining $m = 0$ state. Since the $j = 0$, $m = 0$ state must be orthogonal to the $j = 1$, $m = 0$ state, we must have, up to a phase,

$$|j = 0, m = 0\rangle = \frac{1}{\sqrt{2}} \left(| -\frac{1}{2}, \frac{1}{2}\rangle - |\frac{1}{2}, -\frac{1}{2}\rangle \right). \tag{20.33}$$

The $\sqrt{2}$ is to guarantee that this is a unit vector,

$$\begin{aligned} \langle j = 0, m = 0|j = 0, m = 0\rangle &= \frac{1}{2} \left[\langle -\frac{1}{2}, \frac{1}{2}| - \langle\frac{1}{2}, -\frac{1}{2}| \right] \left[| -\frac{1}{2}, \frac{1}{2}\rangle - |\frac{1}{2}, -\frac{1}{2}\rangle \right] \\ &= \frac{1}{2}[1 + 1] = 1. \end{aligned} \tag{20.34}$$

Thus the four states of two independent spin-1/2 systems correspond to the three states of $j = 1$ plus the single state of $j = 0$.

Let us henceforth simplify the notation, as we did last semester. For the spin-1/2 systems, only the signs of m_i matter, so we will denote

$$| \pm\frac{1}{2}, \pm\frac{1}{2}\rangle = |\pm, \pm\rangle. \tag{20.35}$$

Thus the three $j = 1$ states constructed from two spin 1/2's are

$$|1,1\rangle = |+,+\rangle, \tag{20.36a}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left[|-,+\rangle + |+,-\rangle\right], \tag{20.36b}$$

$$|1,-1\rangle = |-,-\rangle. \tag{20.36c}$$

Note that once one phase is specified, the others are determined by the lowering operator. Note that all three of these states are symmetrical under $1 \leftrightarrow 2$. The one remaining state,

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left[|-,+\rangle - |+,-\rangle\right], \tag{20.37}$$

which is a unit vector, orthogonal to the other states, is antisymmetrical under the interchange $1 \leftrightarrow 2$.

As an example of overkill, let's consider what happens when we apply J_- to the $|0,0\rangle$ state:

$$\begin{aligned}\frac{1}{\hbar}J_-|0,0\rangle &= \left[\frac{1}{2}(\sigma_x - i\sigma_y)_1 + \frac{1}{2}(\sigma_x - i\sigma_y)_2\right]\frac{1}{\sqrt{2}}\left[|-+\rangle - |+-\rangle\right]\\ &= \frac{1}{\sqrt{2}}\left[-|--\rangle + |--\rangle\right] = 0,\end{aligned} \tag{20.38}$$

which indeed demonstrates that $|0,0\rangle$ is the $j = 0$ state.

Recall that last semester, we considered building up spin 1 from two spin 1/2's, but then we did not have the machinery of quantum mechanics. Consider two quantization directions, z and z', which make an angle θ with respect to each other, so we can consider two situations:

1. We can construct the state $j = 1$, $m = 1$ by adding $m_1 = 1/2$, $m_2 = 1/2$, where the spin projections refer to the z axis:
$$J'_z = 1: \quad |11z\rangle = |+z,+z\rangle. \tag{20.39}$$
2. We can construct the state $j = 1$, $m = 1$ by adding $m_1 = 1/2$, $m_2 = 1/2$, where the spin projections refer to the z' axis:
$$J'_{z'} = 1: \quad |11z'\rangle = |+z',+z'\rangle. \tag{20.40}$$

Since the probability of finding $m_1 = 1/2$ in the z' direction given that the state was initially prepared in the state $m_1 = 1/2$ in the z direction is $\cos^2\theta/2$, the probability of finding $m = 1$ in the z direction given that $m = 1$ in the z' direction is

$$\cos^2\theta/2\cos^2\theta/2 = \cos^4\theta/2. \tag{20.41}$$

[Recall Eq. (4.40), for example.] This is in fact correct. On the other hand, we might anticipate that the probability of finding $m = 0$ in z given that the state was prepared with $m = 1$ in z' is

$$\cos^2\theta/2\sin^2\theta/2 = \frac{1}{4}\sin^2\theta, \tag{20.42}$$

which is not correct; it should be a factor of 2 larger, as we saw in Problem 3-1 from last semester. Now we see that the vectors in quantum mechanics are combined in a very definite way, which supplies the missing factor of 2. The probability is the square of a probability amplitude,

$$p(+1z,+1z') = |\langle +1z| + 1z'\rangle|^2, \tag{20.43}$$

so we have interference between waves. It helps to think of

$$|++\rangle = |+\rangle_1|+\rangle_2, \tag{20.44}$$

where for example $|+\rangle_1$ corresponds to putting the first system into the state $m_1 = +1/2$. The states labeled 1 and 2 are independent of each other. Recall the wavefunctions we computed, for example, in Eq. (7.26),

$$\psi_{+z'}(+) = \langle +z| + z'\rangle == e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}, \quad \psi_{+z'}(-) = \langle -z| + z'\rangle = e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}. \tag{20.45}$$

Therefore, up to a phase,

$$\langle +1z| + 1z'\rangle = \langle +1/2z| + 1/2z'\rangle_1\langle +1/2z| + 1/2z'\rangle_2 = \cos\theta/2\cos\theta/2 = \cos^2\theta/2, \tag{20.46}$$

so

$$p(+1z,+1z') = \cos^4\theta/2, \tag{20.47}$$

as anticipated in the simple picture. The real question is what is

$$p(0z,+1z') = |\langle 0z| + 1z'\rangle|^2. \tag{20.48}$$

So we have to work out the probability amplitude

$$\begin{aligned}\langle 0z| + 1z'\rangle &= \frac{1}{\sqrt{2}}\left[\langle -z|_1\langle +z|_2 + \langle +z|_1\langle -z|_2\right]| + z'\rangle_1| + z'\rangle_2 \\ &= \frac{1}{\sqrt{2}}\left[\langle -z| + z'\rangle_1\langle +z| + z'\rangle_2 + \langle +z| + z'\rangle_1\langle -z| + z'\rangle_2\right] \\ &= \frac{1}{\sqrt{2}}\left[\sin\theta/2\cos\theta/2 + \cos\theta/2\sin\theta/2\right] = \sqrt{2}\cos\theta/2\sin\theta/2.\end{aligned} \tag{20.49}$$

Here we ignored a common phase which does not contribute to the probability. The two terms appearing here are identical—they interfere *constructively.* Note that the 1,2 labels have no meaning on the individual amplitudes, $\langle -z| + z'\rangle$ is the same whether it refers to 1 or 2. Thus the probability is, correctly,

$$p(0z,+1z') = 2\cos^2\theta/2\sin^2\theta/2. \tag{20.50}$$

If we were to calculate $\langle 0z, j = 0| + 1z', j = 1\rangle$ in the same way we would get zero—quantum interference is decisive:

$$\langle 00z|11z'\rangle = \frac{1}{\sqrt{2}}\left[\langle -z|_1\langle +z|_2 - \langle +z|_1\langle -z|_2\right]| + z'\rangle_1| + z'\rangle_2$$

m	Number of states	j	(m_1, m_2)
3	1	3	(1,2)
2	2	3,2	$(1,1),(0,2)$
1	3	3, 2, 1	$(1,0),(0,1),(-1,2)$
0	3	3, 2, 1	$(1,-1),(0,0),(-1,1)$
-1	3	3, 2, 1	$(1,-2),(0,-1),(-1,0)$
-2	2	3, 2	$(0,-2),(-1,-1)$
-3	1	3	$(-1,-2)$

Table 20.1: The 15 states that can be constructed by adding spin 1 to spin 2. Shown are the total magnetic quantum numbers, the number of states, the total angular momentum quantum numbers, and the constituent magnetic quantum numbers.

$$= \frac{1}{\sqrt{2}}\left[\langle -z|+z'\rangle_1\langle +z|+z'\rangle_2 - \langle +z|+z'\rangle_1\langle -z|+z'\rangle_2\right]$$
$$= \frac{1}{\sqrt{2}}\left[\sin\theta/2\cos\theta/2 - \cos\theta/2\sin\theta/2\right] = 0. \qquad (20.51)$$

What we have just done is combine four states, those of two independent spin 1/2 systems, to get 4 states, three of spin $j = 1$ and one of spin $j = 0$. How does this generalize? As an example, let's combine two angular momenta

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2, \qquad (20.52)$$

for $j_1 = 1$ and $j_2 = 2$. Table 20.1 shows how these can be combined. The largest value of m is 3, so the largest j value is 3. There are 7 states with

$$j = 3: \quad m = 3, 2, 1, 0, -1, -2, -3. \qquad (20.53)$$

There is a second $m = 2$ state, so there must also be a $j = 2$ set of states, which has 5 members:

$$j = 2: \quad m = 2, 1, 0, -1, -2. \qquad (20.54)$$

There is still a remaining $m = 1$ state. so there must be a set of $j = 1$ states, with 3 members,

$$j = 1: \quad m = 1, 0, -1. \qquad (20.55)$$

This accounts for all the states,

$$(2j_1 + 1)(2j_2 + 1) = 2 \times 5 = 15 = \sum_{j=1}^{3}(2j + 1) = 7 + 5 + 3. \qquad (20.56)$$

Adding angular momenta 1 and 2 give states with angular momenta 1, 2, and 3.

How does it go in general? Consider arbitrary j_1 and j_2. For definiteness, let's assume $j_1 \leq j_2$. The pattern is shown in Table 20.2. So, in general, the

m	Number of states	j	(m_1, m_2)
$j_1 + j_2$	1	$j_1 + j_2$	(j_1, j_2)
$j_1 + j_2 - 1$	2	$j_1 + j_2, j_1 + j_2 - 1$	$(j_1, j_2 - 1), (j_1, j_2 - 1)$
		$\dots$	
$j_2 - j_1$	$2j_1 + 1$	$j_1 + j_2, j_1 + j_2 - 1, \dots j_2 - j_1$	$(j_1, j_2 - 2j_1), \dots, (-j_1, j_2)$
		$\dots$	
$-j_1 - j_2 + 1$	2	$j_1 + j_2, j_1 + j_2 - 1$	$(-j_1, -j_2 + 1), (-j_1 + 1, -j_2)$
$-j_1 - j_2$	1	$j_1 + j_2$	$(-j_1, -j_2)$

Table 20.2: The $(2j_1+1)(2j_2+1)$ states that can be constructed by adding spin j_1 to spin j_2. (We assume $j_2 \geq j_1$.) Shown are the total magnetic quantum numbers, the number of states, the total angular momentum quantum numbers, and the constituent magnetic quantum numbers.

possible values of j are

$$j = j_1 + j_2, j_1 + j_2 - 1, \dots, |j_1 - j_2|. \tag{20.57}$$

The different possible angular momenta differ by unit steps, although the total angular momentum can be an integer or an integer plus 1/2. Check the counting: The number of states is

$$(2j_1 + 1)(2j_2 + 1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j + 1). \tag{20.58}$$

Since this is a sum of a linear function of j it is equal to the number of terms times the average term. Suppose again $j_2 \geq j_1$. Then the number of terms is $2j_1 + 1$, and the average term is $2 \times \frac{1}{2}(j_1 + j_2 + j_2 - j_1) + 1 = 2j_2 + 1$, which verifies the summation.

We can derive this result more elegantly. Recall in Chapter 4, Sec. 4.3, last semester, we proved

$$\sum_{m=-j}^{j} e^{im\phi} = \frac{\sin(j + 1/2)\phi}{\sin \phi/2}. \tag{20.59}$$

A direct proof is

$$\begin{aligned}\sum_{m=-j}^{j} e^{im\phi} &= e^{-ij\phi} \sum_{j=0}^{2j} e^{im\phi} = \frac{e^{-ij\phi}}{1 - e^{i\phi}} \left(1 - e^{i(2j+1)\phi}\right) \\ &= \frac{e^{-i(j+1/2)\phi} - e^{i(j+1/2)\phi}}{e^{-i\phi/2} - e^{i\phi/2}} = \frac{\sin(j + 1/2)\phi}{\sin \phi/2}.\end{aligned} \tag{20.60}$$

Now if we multiply two such summations together, for spin j_1 and j_2, we have

$$\left(\sum_{m_1=-j_1}^{j_1} e^{im_1\phi}\right)\left(\sum_{m_2=-j_2}^{j_2} e^{im_2\phi}\right) = \frac{\sin(j_1 + 1/2)\phi}{\sin \phi/2} \frac{\sin(j_2 + 1/2)\phi}{\sin \phi/2}. \tag{20.61}$$

Now the left side of this equation can be written as $\sum_{m_1 m_2} e^{i(m_1+m_2)\phi}$, since $m = m_1 + m_2$. Using

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)], \tag{20.62}$$

the right side of Eq. (20.61) is

$$\frac{\cos(j_1 - j_2)\phi - \cos(j_1 + j_2 + 1)\phi}{2\sin^2\phi/2}. \tag{20.63}$$

Now

$$\begin{aligned}\sum_{j=j_-}^{j_+} \sin(j+1/2)\phi &= \operatorname{Im}\left[\sum_{j=0}^{j_+} e^{i(j+1/2)\phi} - \sum_{j=0}^{j_- -1} e^{i(j+1/2)\phi}\right] \\ &= \operatorname{Im}\frac{e^{i\phi/2}}{1-e^{i\phi}}\left[\left(1 - e^{i(j_+ +1)\phi}\right) - \left(1 - e^{ij_-\phi}\right)\right] = \operatorname{Im}\frac{e^{ij_-\phi} - e^{i(j_+ +1)\phi}}{-2i\sin\phi/2} \\ &= \frac{\cos j_-\phi - \cos(j_+ + 1)\phi}{2\sin\phi/2}. \end{aligned} \tag{20.64}$$

Thus we conclude from Eqs. (20.59) and (20.63) that

$$\sum_{m_1 m_2} e^{im\phi} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} e^{im\phi}. \tag{20.65}$$

That is, the states with j_1, m_1, j_2, m_2, with

$$-j_1 \le m_1 \le j_1, \quad -j_2 \le m_2 \le j_2, \tag{20.66}$$

can be equally well classified by j and m, where $m = m_1 + m_2$ and $-j \le m \le j$ where

$$|j_1 - j_2| \le j \le j_1 + j_2. \tag{20.67}$$

Putting $\phi = 0$ in Eq. (20.65) gives the sum (20.58) again:

$$(2j_1+1)(2j_2+1) = \sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1). \tag{20.68}$$