Chapter 18

Dynamical Variables and Time Evolution

Consider, for example, the angular momentum,

$$\mathbf{J} = \mathbf{R} \times \mathbf{P} + \mathbf{S}. \tag{18.1}$$

In general, $\mathbf{R}$, $\mathbf{P}$, $\mathbf{S}$ change in time; these are examples of *dynamical variables.*

Let $v(t)$ be a dynamical variable (a particular one, or the whole class of dynamical variables). Under the displacement of the time origin,

$$\bar{t} = t - \delta t, \quad U = 1 + \frac{i}{\hbar}(-\delta t H), \tag{18.2}$$

the new function of the new time equals the old function at the old time:

$$v(t) = \bar{v}(\bar{t}) = \bar{v}(t - \delta t), \tag{18.3}$$

or, by relabeling $t \to t + \delta t$,

$$\bar{v}(t) = v(t + \delta t). \tag{18.4}$$

The only thing that matters is the relative time displacement. The quantum mechanical version of this is generally

$$\bar{X} = U^{-1} X U, \quad U = 1 + \frac{i}{\hbar} G, \tag{18.5}$$

or

$$\bar{X} = X - \delta X, \quad \delta X = \frac{1}{i\hbar}[X, G]. \tag{18.6}$$

On the other hand, by Taylor expanding Eq. (18.4) we have

$$\bar{v}(t) = v(t) + \delta t \frac{d}{dt} v(t) = v(t) - \delta v(t), \tag{18.7}$$

so

$$\delta v(t) = -\delta t \frac{d}{dt} v(t) = \frac{1}{i\hbar}[v(t), -\delta t H], \tag{18.8}$$

or

$$\frac{d}{dt} v(t) = \frac{1}{i\hbar}[v(t), H]. \tag{18.9}$$

The energy operator, or Hamiltonian, describes how the system evolves in time.

Suppose we have a function $F(v(t), t)$ which involves a dynamical variable as well as involving the time explicitly. We have seen an example of this in the boost generator,

$$\mathbf{N} = \mathbf{P}t - M\mathbf{R}. \tag{18.10}$$

Under the unitary time-evolution operator, the function changes:

$$\bar{F} = U^{-1} F(v(t), t) U = F(U^{-1} v(t) U, t) = F(\bar{v}(t), t) = F(v(t), t) - \frac{1}{i\hbar}[F, -\delta t H], \tag{18.11}$$

so

$$\frac{F(\bar{v}(t), t) - F(v(t), t)}{\delta t} = \frac{1}{i\hbar}[F, H]. \tag{18.12}$$

The left-hand side of this equation means, in the limit $\delta t \to 0$

$$\frac{F(v(t+\delta t), t) - F(v(t), t)}{\delta t} \to \frac{d}{dt} F(v(t), t) - \frac{\partial}{\partial t} F(v(t), t), \tag{18.13}$$

where the total derivative acts on both $v(t)$ and t, while the partial derivative removes that part of the time derivative which comes from the explicit appearance of t. Thus we obtain the general formula, for a function of a dynamical variable,

$$\frac{d}{dt} F = \frac{\partial}{\partial t} F + \frac{1}{i\hbar}[F, H], \tag{18.14}$$

which generalizes the equation (18.9) for the time evolution of a dynamical variable.

As an example, consider the momentum [see Eq. (17.64a)],

$$\frac{d}{dt} \mathbf{P}(t) = \frac{1}{i\hbar}[\mathbf{P}(t), H] = 0, \tag{18.15}$$

because there is no explicit appearance of t, because displacements make no reference to time. This states that momentum is *conserved*. (Implicitly, we are considering the whole system with no external forces; a completely described isolated system.) The angular momentum statement is similar, from Eq. (17.64b)

$$\frac{d}{dt} \mathbf{J}(t) = \frac{1}{i\hbar}[\mathbf{J}(t), H] = 0, \tag{18.16}$$

or *angular momentum is conserved*. What about boosts, where, as noted above, t appears explicitly? From Eq. (17.64c),

$$\frac{d}{dt} \mathbf{N}(t) = \frac{\partial}{\partial t} \mathbf{N}(\mathbf{t}) + \frac{1}{i\hbar}[\mathbf{N}(t), H] = \mathbf{P} + \frac{1}{i\hbar}[\mathbf{N}, H] = 0, \tag{18.17}$$

so $\mathbf{N}$ is also a constant in time.

$$\frac{d}{dt}\mathbf{N} = 0, \quad \mathbf{N}(t) = \mathbf{P}(t)t - M\mathbf{R}(t). \tag{18.18}$$

How does this happen? We know $\mathbf{P}$ is constant, so

$$0 = \frac{d\mathbf{N}}{dt} = \mathbf{P} - M\frac{d\mathbf{R}(t)}{dt}, \tag{18.19}$$

or

$$\mathbf{P} = M\mathbf{V}(t), \quad \mathbf{V}(t) = \frac{d\mathbf{R}(t)}{dt}, \tag{18.20}$$

in terms of the velocity of the system. Thus the velocity $\mathbf{V}$ (of the center of mass) is a constant, $\mathbf{P}/M$. Again the identification of M with mass is seen. Thus we conclude

$$\frac{\mathbf{P}}{M} = \frac{d}{dt}\mathbf{R}(t) = \frac{1}{i\hbar}[\mathbf{R}(t), H]. \tag{18.21}$$

Therefore, this last gives information about the structure of the energy operator.

As noted above $\mathbf{R}$, $\mathbf{P}$ are like q and p three times over, where recall that for a finite displacement

$$e^{-iq'p}qe^{iq'p} = q - q', \tag{18.22}$$

so analogously,

$$e^{-i\mathbf{R}'\cdot\mathbf{P}/\hbar}\mathbf{R}e^{i\mathbf{R}'\cdot\mathbf{P}/\hbar} = \mathbf{R} - \mathbf{R}'. \tag{18.23}$$

As in homework, we prove this directly by differentiation with respect to $\mathbf{R}'$:

$$\frac{\partial}{\partial R'_l}\left(e^{-i\mathbf{R}'\cdot\mathbf{P}/\hbar}R_k e^{i\mathbf{R}'\cdot\mathbf{P}/\hbar}\right) = e^{-i\mathbf{R}'\cdot\mathbf{P}/\hbar}\frac{i}{\hbar}[R_k, P_l]e^{i\mathbf{R}'\cdot\mathbf{P}/\hbar} = -\delta_{kl} = -\frac{\partial R'_k}{\partial R'_l}, \tag{18.24}$$

because the various components of $\mathbf{P}$ commute,

$$[P_k, P_l] = 0. \tag{18.25}$$

Supplying the constant of integration, we obtain the expected result,

$$e^{-i\mathbf{R}'\cdot\mathbf{P}/\hbar}\mathbf{R}e^{i\mathbf{R}'\cdot\mathbf{P}/\hbar} = \mathbf{R} - \mathbf{R}'. \tag{18.26}$$

Similarly, the counterpart to

$$e^{ip'q}pe^{-ip'q} = p - p' \tag{18.27}$$

is

$$e^{i\mathbf{P}'\cdot\mathbf{R}/\hbar}\mathbf{P}e^{-i\mathbf{P}'\cdot\mathbf{R}/\hbar} = \mathbf{P} - \mathbf{P}', \tag{18.28}$$

because the different components of $\mathbf{R}$ commute,

$$[R_k, R_l] = 0. \tag{18.29}$$

Then it follows for any function of $\mathbf{R}$ and $\mathbf{P}$ that

$$e^{i\mathbf{P}'\cdot\mathbf{R}/\hbar}F(\mathbf{R}, \mathbf{P})e^{-i\mathbf{P}'\cdot\mathbf{R}/\hbar} = F(\mathbf{R}, \mathbf{P} - \mathbf{P}'). \tag{18.30}$$

Let's go to the infinitesimal limit, $\mathbf{P}' \to \delta\mathbf{P}'$,

$$e^{i\delta\mathbf{P}'\cdot\mathbf{R}/\hbar}F(\mathbf{R},\mathbf{P})e^{-i\delta\mathbf{P}'\cdot\mathbf{R}/\hbar} = F(\mathbf{R},\mathbf{P}-\delta\mathbf{P}'), \tag{18.31}$$

or

$$F+\frac{i}{\hbar}[\delta\mathbf{P}'\cdot\mathbf{R},F] = F-\delta\mathbf{P}'\cdot\frac{\partial F}{\partial\mathbf{P}'} = F-\delta P'_x\frac{\partial F}{\partial P_x}-\delta P'_y\frac{\partial F}{\partial P_y}-\delta P'_z\frac{\partial F}{\partial P_z}, \tag{18.32}$$

or succinctly,

$$\frac{1}{i\hbar}[\mathbf{R},F] = \frac{\partial F}{\partial\mathbf{P}}. \tag{18.33}$$

This formula is an immediate generalization of

$$\frac{1}{i\hbar}[R_k,P_l] = \delta_{kl}, \quad [R_k,R_l] = 0. \tag{18.34}$$

In fact if $F = P_l$, this equation says

$$\frac{1}{i\hbar}[R_k,P_l] = \frac{\partial P_l}{\partial P_k} = \delta_{kl}, \tag{18.35}$$

while if $F = R_l$,

$$\frac{1}{i\hbar}[R_k,R_l] = \frac{\partial R_l}{\partial P_k} = 0. \tag{18.36}$$

In the same way, from Eq. (18.26), we can show

$$\frac{1}{i\hbar}[\mathbf{P},F] = -\frac{\partial F}{\partial\mathbf{R}}, \tag{18.37}$$

where again if $F = R_l$,

$$\frac{1}{i\hbar}[P_k,R_l] = -\frac{\partial R_l}{\partial R_k} = -\delta_{kl}, \tag{18.38}$$

and if $F = P_l$,

$$\frac{1}{i\hbar}[P_k,P_l] = -\frac{\partial P_l}{\partial R_k} = 0. \tag{18.39}$$

Return at last to Eq. (18.21),

$$\frac{\mathbf{P}}{M} = \frac{d\mathbf{R}}{dt} = \frac{1}{i\hbar}[\mathbf{R}(t),H] = \frac{\partial H}{\partial\mathbf{P}}, \tag{18.40}$$

which looks like Hamilton's equation. This implies that

$$H = \frac{\mathbf{P}^2}{2M} + H_{\text{int}}, \tag{18.41}$$

where

$$\frac{\partial H_{\text{int}}}{\partial\mathbf{P}} = 0. \tag{18.42}$$

H_{int} is called the internal energy, whereas $\mathbf{P}^2/2M$ is the part of the energy referring to the motion of the system as a whole. The momenta also advance in time at a known rate,

$$\frac{d\mathbf{P}}{dt} = 0 = \frac{1}{i\hbar}[\mathbf{P}, H] = \frac{1}{i\hbar}[\mathbf{P}, H_{\text{int}}], \tag{18.43}$$

or

$$0 = \frac{d\mathbf{P}}{dt} = -\frac{\partial H_{\text{int}}}{\partial \mathbf{R}}, \tag{18.44}$$

which is the other set of Hamilton's equations. These results say that H_{int} does not depend on either $\mathbf{R}$ or $\mathbf{P}$.

Classical mechanics is derived by taking $\hbar \to 0$, so that the quantum of action is negligible, operator properties are irrelevant). But we see that Hamilton's equations hold in general.