Chapter 16

Translations

The time has come to go back and talk about displacement. Recall the rigid displacement of the origin of the coordinate system pictured in Fig. 16.1. The origin is displaced by an infinitesimal amount $\delta\boldsymbol{\epsilon}$, so the new coordinates of a point are related to the old coordinates of the same point by

$$\bar{\mathbf{r}} = \mathbf{r} - \delta\boldsymbol{\epsilon}. \tag{16.1}$$

Correspondingly, there is an infinitesimal unitary transformation

$$U = 1 + \frac{i}{\hbar}G, \quad G = \delta\boldsymbol{\epsilon}\cdot\mathbf{P}, \tag{16.2}$$

where the vector $\mathbf{P}$ is the momentum operator. Since $\mathbf{P}$ is a vector, it must satify the following communtation relation with the angular momentum,

$$\frac{1}{i\hbar}[\mathbf{P}, \delta\boldsymbol{\omega}\cdot\mathbf{J}] = \delta\boldsymbol{\omega}\times\mathbf{P}, \tag{16.3}$$

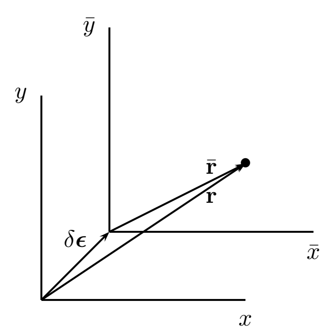

Figure 16.1: A point in space as described in two different coordinate systems, O and $\bar{O}$, where the latter's origin is displaced from the former's by an amount $\delta\boldsymbol{\epsilon}$.

since this is how any vector transforms under a rotation. Turn this around to see the effect of **P** on **J**:

$$\frac{1}{i\hbar}[\delta\boldsymbol{\epsilon}\cdot\mathbf{P},\delta\boldsymbol{\omega}\cdot\mathbf{J}]=\delta\boldsymbol{\epsilon}\cdot(\delta\boldsymbol{\omega}\times\mathbf{P}). \tag{16.4}$$

The question we are asking is, what is the effect of a displacement on **J**? Because

$$\delta\boldsymbol{\epsilon}\cdot(\delta\boldsymbol{\omega}\times\mathbf{P})=(\delta\epsilon\times\delta\boldsymbol{\omega})\cdot\mathbf{P}=-(\delta\boldsymbol{\omega}\times\delta\boldsymbol{\epsilon})\cdot\mathbf{P}=-\delta\boldsymbol{\omega}\cdot(\delta\boldsymbol{\epsilon}\times\mathbf{P}), \tag{16.5}$$

or

$$\frac{1}{i\hbar}[\mathbf{J},\delta\boldsymbol{\epsilon}\cdot\mathbf{P}]=\delta\boldsymbol{\epsilon}\times\mathbf{P}, \tag{16.6}$$

in which we see appear the moment of momentum–when the origin in moved, the moment changes. This suggests introducing a position operator **R**, and writing

$$\mathbf{J}=\mathbf{R}\times\mathbf{P}+\mathbf{S}, \tag{16.7}$$

where the first term is the orbital angular momentum and the second is the spin. Since under a displacement,

$$\frac{1}{i\hbar}[\mathbf{R},\delta\boldsymbol{\epsilon}\cdot\mathbf{P}]=\delta\boldsymbol{\epsilon}, \tag{16.8}$$

because this is just how the position vector changes under a displacement, while

$$\frac{1}{i\hbar}[\mathbf{P},\delta\boldsymbol{\epsilon}\cdot\mathbf{P}]=0, \tag{16.9}$$

because momentum shouldn't change under a displacement, and finally,

$$\frac{1}{i\hbar}[\mathbf{S},\delta\boldsymbol{\epsilon}\cdot\mathbf{P}]=0, \tag{16.10}$$

because spin has nothing to do with position, we see that Eq. (16.6) is automatically satisfied. Moreover, the orbital angular momentum satisfies the commutation relations of angular momentum: That is, if

$$\mathbf{L}=\mathbf{R}\times\mathbf{P}, \tag{16.11}$$

then

$$\frac{1}{i\hbar}[\mathbf{L},\delta\boldsymbol{\omega}\cdot\mathbf{L}]=\delta\boldsymbol{\omega}\times\mathbf{L}. \tag{16.12}$$

[Proof: Homework.] The spin in general must be present because orbital angular momentum can only take integral values of l,

$$\mathbf{L}^{2\prime}=l(l+1)\hbar^2,\quad L_z'=m\hbar,\quad -l\le m_l\le l, \tag{16.13}$$

whereas angular momentum can take on half-integral as well as integral values.

Thus we have derived

$$\frac{1}{i\hbar}[R_k,P_l]=\delta_{kl}, \tag{16.14}$$

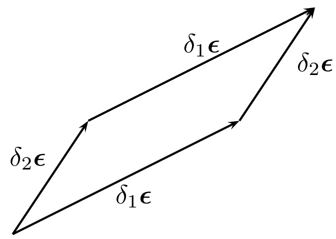

Figure 16.2: Parallelogram formed by considering two successive translations, first by an amount $\delta_1\boldsymbol{\epsilon}$ followed by $\delta_2\boldsymbol{\epsilon}$, or in the other order, $\delta_2\boldsymbol{\epsilon}$ followed by $\delta_1\boldsymbol{\epsilon}$. The net result is the same.

which generalizes the Heisenberg commutation relations we studied above,

$$\frac{1}{i\hbar}[q,p]=1, \tag{16.15}$$

to three independent sets of q, p variables.

Now why is it that the momentum operators commute as shown in Eq. (16.9)? The physical or geometrical reason is that translations commute with each other, as shown in Fig. 16.2. The figure shows that the order of doing translations does not matter. In contrast, rotations are non-commutative: a rotation about the x axis followed by one about the y axis is not the same as a rotation about the y axis followed by a rotation about the x axis. The two rotations differ by a rotation about the z axis, which is the content of the commutation relation

$$J_xJ_y-J_yJ_x=i\hbar J_z. \tag{16.16}$$

Let's check this analytically. If we do the translations in the first order,

$$\bar{\mathbf{r}}_1=\mathbf{r}-\delta_1\boldsymbol{\epsilon},\quad \bar{\mathbf{r}}_2=\mathbf{r}_1-\delta_2\boldsymbol{\epsilon}=\mathbf{r}-(\delta_1\boldsymbol{\epsilon}+\delta_2\boldsymbol{\epsilon}), \tag{16.17a}$$

while if we do them in the opposite order,

$$\bar{\mathbf{r}}_2=\mathbf{r}-\delta_2\boldsymbol{\epsilon},\quad \bar{\mathbf{r}}_1=\mathbf{r}_2-\delta_1\boldsymbol{\epsilon}=\mathbf{r}-(\delta_2\boldsymbol{\epsilon}+\delta_1\boldsymbol{\epsilon}), \tag{16.17b}$$

so these are the same. The order of translations makes no difference, or

$$[\delta_1\boldsymbol{\epsilon}\cdot\mathbf{P},\delta_2\boldsymbol{\epsilon}\cdot\mathbf{P}]=0, \tag{16.18}$$

or in components,

$$[P_k,P_l]=0. \tag{16.19}$$

This says that all components of momentum are compatible; unlike J_x, J_y, and J_z, we can specify values for P_x, P_y, P_z simultaneously.

Contrast the above calculation with that for successive rotations:

$$\bar{\mathbf{r}}_1 = \mathbf{r}-\delta_1\boldsymbol{\omega}\times\mathbf{r}, \tag{16.20a}$$

$$\bar{\mathbf{r}}_{21} = \mathbf{r}_1-\delta_2\boldsymbol{\omega}\times\mathbf{r}_1=\mathbf{r}-(\delta_1\boldsymbol{\omega}+\delta_2\boldsymbol{\omega})\times\mathbf{r}+\delta_2\boldsymbol{\omega}\times(\delta_1\omega\times\mathbf{r}), \tag{16.20b}$$

while if we do them in the opposite order,

$$\bar{\mathbf{r}}_2 = \mathbf{r} - \delta_2\boldsymbol{\omega} \times \mathbf{r}, \tag{16.20c}$$

$$\bar{\mathbf{r}}_{12} = \mathbf{r}_2 - \delta_1\boldsymbol{\omega} \times \mathbf{r}_2 = \mathbf{r} - (\delta_2\boldsymbol{\omega} + \delta_1\boldsymbol{\omega}) \times \mathbf{r} + \delta_1\boldsymbol{\omega} \times (\delta_2\boldsymbol{\omega} \times \mathbf{r}). \tag{16.20d}$$

Here we have kept linear and bilinear terms in the rotations. These two rotations are not the same:

$$\delta_{[12]}\mathbf{r} = \mathbf{r}_{12} - \mathbf{r}_{21} = \delta_1\boldsymbol{\omega} \times (\delta_2\boldsymbol{\omega} \times \mathbf{r}) - \delta_2\boldsymbol{\omega} \times (\delta_1\boldsymbol{\omega} \times \mathbf{r}) = (\delta_1\boldsymbol{\omega} \times \delta_2\boldsymbol{\omega}) \times \mathbf{r}, \tag{16.21}$$

which is a rotation about the direction perpendicular to both $\delta_1\boldsymbol{\omega}$ and $\delta_2\boldsymbol{\omega}$.

The commutation relation (16.14) says that perpendicular components of $\mathbf{R}$ and $\mathbf{P}$ commute with each other,

$$\frac{1}{i\hbar}[R_k, P_l] = 0, \quad k \neq l, \tag{16.22}$$

whereas parallel components fail to commute,

$$\frac{1}{i\hbar}[R_k, P_k] = 1, \tag{16.23}$$

because the kth component of position changes by $\delta\epsilon_k$ if the coordinate system is displaced in the k direction by that amount. This says that perpendicular components of $\mathbf{R}, \mathbf{P}$ are compatible, while parallel comments of $\mathbf{R}, \mathbf{P}$ are incompatible. So that means that we cannot measure R_x, P_x simultaneously, as we saw at the very beginning of the course.

As noted above, the spin is unaffected by translation,

$$\delta S_k = 0 = \frac{1}{i\hbar}[S_k, \delta\boldsymbol{\epsilon} \cdot \mathbf{P}], \tag{16.24}$$

that is, $\mathbf{S}, \mathbf{P}$ are compatible,

$$[S_k, P_k] = 0. \tag{16.25}$$

Now the new vector $\mathbf{R}$ must transform appropriately under a rotation,

$$\begin{aligned}\delta_{\delta\boldsymbol{\omega}}\mathbf{R} &= \delta\boldsymbol{\omega} \times \mathbf{R} = \frac{1}{i\hbar}[\mathbf{R}, \delta\boldsymbol{\omega} \cdot \mathbf{J}] \\ &= \frac{1}{i\hbar}[\mathbf{R}, \delta\boldsymbol{\omega} \cdot (\mathbf{R} \times \mathbf{P}) + \delta\boldsymbol{\omega} \cdot \mathbf{S}] \\ &= \frac{1}{i\hbar}[\mathbf{R}, (\delta\boldsymbol{\omega} \times \mathbf{R}) \cdot \mathbf{P} + \delta\boldsymbol{\omega} \cdot \mathbf{S}]. \end{aligned} \tag{16.26}$$

The first commutator here looks like Eq. (16.8), by regarding $\delta\boldsymbol{\omega} \times \mathbf{R} = \delta\boldsymbol{\epsilon}$, and if this commutes with $\mathbf{R}$, we conclude from Eq. (16.8) that this commutation relation is satisfied if

$$[R_k, S_l] = 0, \tag{16.27}$$

which as we would expect, since $\mathbf{S}$ has nothing to do with either position or momentum. But in fact $\delta\boldsymbol{\omega} \times \mathbf{R}$ is an operator, so this only works if $\delta\boldsymbol{\omega} \times \mathbf{R}$ commutes with $\mathbf{R}$, or

$$[R_k, R_l] = 0. \tag{16.28}$$

Put this all together:

$$[R_k, R_l] = 0, \tag{16.29a}$$

$$[P_k, P_l] = 0, \tag{16.29b}$$

$$\frac{1}{i\hbar}[R_k, P_l] = \delta_{kl}, \tag{16.29c}$$

and the spin is independent of the position and momentum,

$$[R_k, S_l] = [P_k, S_l] = 0. \tag{16.30}$$

As for $\mathbf{S}$ it must rotate as a vector, so

$$\delta_{\delta\boldsymbol{\omega}}\mathbf{S} = \delta\boldsymbol{\omega} \times \mathbf{S} = \frac{1}{i\hbar}[\mathbf{S}, \delta\boldsymbol{\omega} \cdot \mathbf{S}], \tag{16.31}$$

so, for example,

$$i\hbar S_z = S_x S_y - S_y S_x, \tag{16.32}$$

or generally,

$$\mathbf{S} \times \mathbf{S} = i\hbar\mathbf{S}. \tag{16.33}$$