Chapter 15

More on harmonic oscillator and angular momentum

The abstract statement

$$|n\rangle = \frac{(y^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{15.1}$$

contains more information than the construction of the excited-state wavefunctions $\psi_n(q')$ in terms of the ground-state wavefunction. We now want to obtain the momentum-space wavefunctions, $\psi_n(p')$. (We could obtain that information, with more work, from the position-space wavefunction.) Recall

$$dq'|\psi_n(q')|^2 \tag{15.2}$$

is the probability of finding q' in the dq' interval; therefore,

$$dp'|\psi_n(p')|^2 \tag{15.3}$$

is the probability of finding p' in the interval between p' and $p' + dp'$. We construct this momentum wavefunction directly from Eq. (15.1), or

$$\psi_n(p') = \langle p'|n\rangle = \langle p'|\frac{(q-ip)^n}{\sqrt{2^n n!}}|0\rangle. \tag{15.4}$$

Now recall for p states

$$\langle p'|p = p'\langle p'|, \quad \langle p'|q = i\frac{\partial}{\partial p'}\langle p'|, \tag{15.5}$$

which differs from

$$\langle q'|q = q'\langle q'|, \quad \langle q'|p = \frac{1}{i}\frac{\partial}{\partial q'}\langle q'| \tag{15.6}$$

by the substitution

$$q \to p, \quad p \to -q. \tag{15.7}$$

Thus, following the previous path,

$$\langle p'|n\rangle = \frac{i^n}{\sqrt{2^n n!}}\left(\frac{\partial}{\partial p'} - p'\right)^n \langle p'|0\rangle, \tag{15.8}$$

The ground-state wavefunction is analogous to that in position-space:

$$\psi_o(p') = \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}p'^2}. \tag{15.9}$$

The latter follows by Fourier transformation (See Problem 2, Assignment 2),

$$\begin{aligned}
\psi_0(p') &= \langle p'|0\rangle = \int_{-\infty}^{\infty} \langle p'|q'\rangle dq' \langle q'|0\rangle = \int_{-\infty}^{\infty} dq' \frac{e^{-iq'p'}}{\sqrt{2\pi}}\psi_0(q') \\
&= \frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq' e^{-\frac{1}{2}q'^2 - iq'p'} \\
&= \frac{1}{\pi^{1/4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq' e^{-\frac{1}{2}(q'+ip')^2} e^{-\frac{1}{2}p'^2} \\
&= \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}p'^2},
\end{aligned} \tag{15.10}$$

where we have noted that [Eq. (14.36)]

$$\int_{-\infty}^{\infty} dq' e^{-\frac{1}{2}(q'+ip')^2} = \sqrt{2\pi}, \tag{15.11}$$

since the integral is invariant under a shift of variable, $q' \to q' - ip'$.

Alternatively, we can obtain the ground-state wavefunction directly, from the annihilation operator statement,

$$y|0\rangle = 0. \tag{15.12}$$

This implies

$$\left(i\frac{\partial}{\partial p'} + ip'\right)\langle p'|0\rangle = 0, \tag{15.13}$$

which has as solution

$$\langle p'|0\rangle = Ce^{-\frac{1}{2}p'^2}. \tag{15.14}$$

We can choose the normalization factor to be $C = \pi^{-1/4}$ in order to satisfy the probability normalization condition

$$\int_{-\infty}^{\infty} dp' |\psi_0(p')|^2 = 1. \tag{15.15}$$

Now, by using the identity (14.52), with $q' \to p'$, we write Eq. (15.8) as

$$\langle p'|n\rangle = \frac{i^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{1}{2}p'^2}\left(\frac{d}{dp'}\right)^n e^{-p'^2}, \tag{15.16}$$

or in terms of Hermite polynomials in momentum [Eq. (14.55) with $q' \to p'$],

$$\psi_n(p') = \frac{(-i)^n}{\sqrt{\pi^{1/2}2^n n!}} H_n(p') e^{-\frac{1}{2}p'^2}. \tag{15.17}$$

We could also obtain this result directly from the position-space wavefunction by Fourier transformation.

All of this came from considering the case when "the angular momentum was nearly aligned with the z axis," or $m \approx j$. What about the opposite situation, when $m \approx -j$? Once again, let us suppose

$$j \gg 1, \tag{15.18}$$

but now let

$$m = -j + n, \quad n = 0, 1, 2, \ldots, \quad \text{or} \quad n = m + j. \tag{15.19}$$

Now when m increases, so does n. Recall the raising operator statement

$$\frac{1}{\hbar} J_+ |jm\rangle = \sqrt{(j-m)(j+m+1)}|jm+1\rangle = \sqrt{(2j-n)(n+1)}|jm+1\rangle, \tag{15.20}$$

or with $j \gg n$

$$\frac{1}{\hbar} J_+ |n\rangle \approx \sqrt{2j}\sqrt{n+1}|n+1\rangle. \tag{15.21}$$

This time define

$$\frac{1}{\hbar} J_+ = \sqrt{2j} y^\dagger, \tag{15.22}$$

so we obtain the following statement which makes no reference to j:

$$y^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle. \tag{15.23}$$

Likewise, from

$$\frac{1}{\hbar} J_- |jm\rangle = \sqrt{(j+m)(j-m+1)}|jm-1\rangle \approx \sqrt{n(2j)}|jm-1\rangle, \tag{15.24}$$

we define

$$\frac{1}{\hbar} J_- = \sqrt{2j} y, \tag{15.25}$$

as the adjoint of Eq. (15.22), where

$$y|n\rangle = \sqrt{n}|n-1\rangle. \tag{15.26}$$

Everything is the same as before, in terms of the oscillator variables, but the construction is different. Previously [Eq. (14.6)]

$$m = j - n, \; n \ll j: \quad \frac{1}{\hbar} J_+ = \sqrt{2j} y, \quad J_- = \sqrt{2j} y^\dagger. \tag{15.27}$$

Both situations give rise to y and $y^\dagger$ with the same properties, which follow from

$$\left[\frac{1}{\hbar}J_+, \frac{1}{\hbar}J_-\right] = 2\frac{1}{\hbar}J_z. \tag{15.28}$$

In the first situation,

$$j \gg 1,\ m \approx j : \quad [y, y^\dagger] = 1, \quad \text{since} \quad \frac{J_z}{\hbar} \approx j, \tag{15.29a}$$

while in the second

$$j \gg 1,\ m \approx -j : \quad [y^\dagger, y] = -1, \quad \text{since} \quad \frac{J_z}{\hbar} \approx -j, \tag{15.29b}$$

or

$$[y, y^\dagger] = 1, \tag{15.29c}$$

which is exactly the same statement. Of course, the commutation relation follows from the effect of y, $y^\dagger$ on any state,

$$\begin{aligned}(yy^\dagger - y^\dagger y)|n\rangle &= y\sqrt{n+1}|n+1\rangle - y^\dagger\sqrt{n}|n-1\rangle = \sqrt{n+1}\sqrt{n+1}|n\rangle - \sqrt{n}\sqrt{n}|n\rangle \\ &= (n+1-n)|n\rangle = |n\rangle.\end{aligned} \tag{15.30}$$

Because this holds for any n state, and these states form a complete set,

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1, \tag{15.31}$$

we must have

$$[y, y^\dagger] = (yy^\dagger - y^\dagger y) = 1. \tag{15.32}$$

Thus we have constructed two different ways of arriving at the oscillator, which can be transcribed in terms of Hermitian variables as

$$y = \frac{q+ip}{\sqrt{2}}, \quad y^\dagger = \frac{q-ip}{\sqrt{2}}, \tag{15.33}$$

so because

$$y^\dagger y = \frac{q^2+p^2}{2} - \frac{1}{2}, \tag{15.34}$$

the eigenvalue condition

$$(y^\dagger y)' = n \tag{15.35}$$

implies the energy eigenvalues

$$\left(\frac{q^2+p^2}{2}\right)' = n + \frac{1}{2}. \tag{15.36}$$

Even though these two oscillator systems describe two extreme limits of angular momentum, together they describe angular momentum in general. Let us define two non-negative integers $n_\pm$ by

$$j - m = n_-, \quad n_- = 0, 1, 2, \ldots, \tag{15.37a}$$

$$j + m = n_+, \quad n_+ = 0, 1, 2, \ldots, \tag{15.37b}$$

so

$$j = \frac{1}{2}(n_+ + n_-), \quad m = \frac{1}{2}(n_+ - n_-). \tag{15.38}$$

We can use $n_\pm$ in place of j and m. Now relabel everything:

$$\frac{1}{\hbar} J_+ |jm\rangle = \sqrt{(j-m)(j+m+1)}|jm+1\rangle \tag{15.39}$$

becomes

$$\frac{1}{\hbar} J_+ |n_+, n_-\rangle = \sqrt{n_-}\sqrt{n_+ + 1}|n_+ + 1, n_- - 1\rangle, \tag{15.40}$$

and

$$\frac{1}{\hbar} J_- |jm\rangle = \sqrt{(j+m)(j-m+1)}|jm-1\rangle \tag{15.41}$$

becomes

$$\frac{1}{\hbar} J_- |n_+, n_-\rangle = \sqrt{n_+}\sqrt{n_- + 1}|n_+ - 1, n_- + 1\rangle. \tag{15.42}$$

Recall

$$y^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle, \quad y|n\rangle = \sqrt{n}|n-1\rangle, \tag{15.43}$$

for the oscillator. Here we see these effects twice over. There are two *independent* types of operators,

$$\sqrt{n_-}\sqrt{n_+ + 1}|n_+ + 1, n_- - 1\rangle = y_+^\dagger y_- |n_+, n_-\rangle. \tag{15.44}$$

Here $y_+^\dagger$ acts only on the n_+ variable, while y_- acts only on the n_- variable. In the oscillator limits, we saw only one of these operators. Similarly, we have

$$\sqrt{n_+}\sqrt{n_- + 1}|n_+ - 1, n_- + 1\rangle = y_-^\dagger y_+ |n_+, n_-\rangle. \tag{15.45}$$

To complete the story, look at

$$\frac{1}{\hbar} J_z |jm\rangle = m|jm\rangle = \frac{1}{2}(n_+ - n_-)|n_+, n_-\rangle. \tag{15.46}$$

Therefore we want to identify

$$y_+^\dagger y_+ |n_+, n_-\rangle = n_+ |n_+, n_-), \quad y_-^\dagger y_- |n_+, n_-\rangle = n_- |n_+, n_-); \tag{15.47}$$

in this way, J_x, J_y, and J_z can be expressed in terms of y_+, $y_+^\dagger$, y_-, $y_-^\dagger$:

$$\frac{1}{\hbar} J_+ = y_+^\dagger y_-, \tag{15.48a}$$

$$\frac{1}{\hbar} J_- = y_-^\dagger y_+, \tag{15.48b}$$

$$\frac{1}{\hbar} J_z = \frac{1}{2}(y_+^\dagger y_+ - y_-^\dagger y_-). \tag{15.48c}$$

We need to hammer home the fact that the $+$, $-$, operators are independent:

$$[y_+, y_+^\dagger] = 1 \tag{15.49a}$$
$$[y_-, y_-^\dagger] = 1 \tag{15.49b}$$
$$[y_-, y_+] = [y_+, y_-^\dagger] = 0. \tag{15.49c}$$

We see the last because

$$y_+ y_-^\dagger |n_+, n_-\rangle = y_+\sqrt{n_- + 1}|n_+, n_- + 1\rangle = \sqrt{n_+}\sqrt{n_- + 1}|n_+ - 1, n_- + 1\rangle, \tag{15.50}$$

while

$$y_-^\dagger y_+ |n_+, n_-\rangle = y_-^\dagger \sqrt{n_+}|n_+ - 1, n_-\rangle = \sqrt{n_+}\sqrt{n_- + 1}|n_+ - 1, n_- + 1\rangle, \tag{15.51}$$

so

$$(y_+ y_-^\dagger - y_-^\dagger y_+)|n_+, n_-\rangle = 0. \tag{15.52}$$

Since any vector can be expressed in terms of the $|n_+, n_-\rangle$ states—they form a complete set—the result (15.49c) follows.

We now want to organize the results in one structure, since J_x, J_y, and J_z are components of the angular momentum vector. Construct a two component vector,

$$y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix}, \quad y^\dagger = (y_+^\dagger, y_-^\dagger). \tag{15.53}$$

We can write the combinations of these operators in terms of 2×2 matrices. For example,

$$y_+^\dagger y_- = \begin{pmatrix} y_+ \\ y_- \end{pmatrix}^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} = (y_+^\dagger, y_-^\dagger) \begin{pmatrix} y_- \\ 0 \end{pmatrix}. \tag{15.54}$$

Similarly,

$$y_-^\dagger y_+ = \begin{pmatrix} y_+ \\ y_- \end{pmatrix}^\dagger \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} = (y_+^\dagger, y_-^\dagger) \begin{pmatrix} 0 \\ y_+ \end{pmatrix}, \tag{15.55}$$

and

$$\frac{1}{2}(y_+^\dagger y_+ - y_-^\dagger y_-) = \begin{pmatrix} y_+ \\ y_- \end{pmatrix}^\dagger \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} = (y_+^\dagger, y_-^\dagger)\frac{1}{2} \begin{pmatrix} y_+ \\ -y_- \end{pmatrix}. \tag{15.56}$$

Now we recognize these matrices in terms of Pauli matrices:

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\sigma_z, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x + i\sigma_y), \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x - i\sigma_y). \tag{15.57}$$

Thus, we have the constructions, in terms of the vectors (15.53)

$$\frac{1}{\hbar}J_+ = \frac{1}{\hbar}(J_x + iJ_y) = y^\dagger \frac{1}{2}(\sigma_x + i\sigma_y)y, \tag{15.58a}$$
$$\frac{1}{\hbar}J_- = \frac{1}{\hbar}(J_x - iJ_y) = y^\dagger \frac{1}{2}(\sigma_x - i\sigma_y)y, \tag{15.58b}$$
$$\frac{1}{\hbar}J_z = y^\dagger \frac{1}{2}\sigma_z y, \tag{15.58c}$$

or as a single vector statement

$$\frac{1}{\hbar}\mathbf{J} = y^\dagger \frac{1}{2}\boldsymbol{\sigma} y. \tag{15.59}$$

The properties of $\mathbf{J}$ follow from those of y and $y^\dagger$. This says that any angular momentum can be built from the addition of spin-1/2 angular momenta. This provides a machinery for combining similar systems. It is an example of a technique called "second quantization," which is used to describe many electrons or photons, for example.