Chapter 12

Compatible Observables

Quantum mechanics is a mathematical transcription of what the experimenter does. In classical mechanics, we represent physical quantities by numbers. In quantum mechanics, physical quantities are represented by noncommuting quantities—reflecting the "graininess" of the atomic world. We can only predict probabilities; the disturbance produced by measurements cannot be made as small as you like—as in the Stern-Gerlach experiment, measurement of one quantity destroys all information about another quantity. The language of quantum mechanics necessarily requires use of complex numbers.

We were recognizing that there was a general scheme in which the doublet system was just one limit, the other being a system with an infinite number of states. Out of quantum mechanics emerges the classical physics in the macroscopic domain. It is a kind of self-consistency, for the measurements which yield the quantum physics are classical.

We have been talking of systems which have only one property. This is not realistic, but was a harmless simplification. Consider the Stern-Gerlach experiment as sketched in Fig. 12.1. In this experiment the magnetic moment in the z direction is measured, μ_z. We generalize this property to a general one A. To have a clean experiment, we want to select atoms with a well defined speed, so we do a v_x selection. Alternatively, we could do the v_x selection after the beam goes through the Stern-Gerlach apparatus. It makes no difference whether v_x is measured before or after μ_z is measured. Thus, we say v_x, μ_z are compatible properties.

We generalize this be considering two compatible properties A_1, A_2; that is they are non-interfering, in that measurement of one does not interfere with measurement of the other. Let's suppose we make a selective measurement of A_1, followed by a selective measurement of A_2, or vice versa. In terms of measurement symbols

$$|a_1'a_2'| = |a_2'a_1'| = |a_1'||a_2'| = |a_2'||a_1'|, \tag{12.1}$$

where a_1' is the value of A_1 selected and a_2' is the value of A_2 selected. What this says is that the order of doing compatible measurements does not matter.

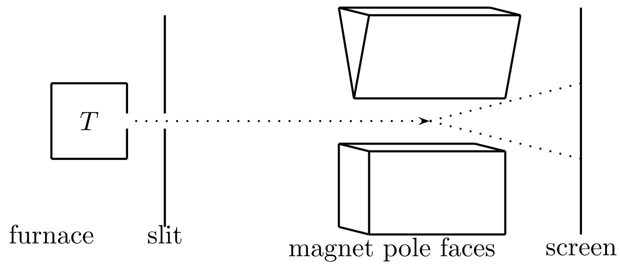

Figure 12.1: Sketch of a Stern-Gerlach experiment. Atoms are heated in the furnace, exit through a small hole, and collimated by a slit in an otherwise opaque screen. The atomic beam then enters an inhomogeneous magnetic field, where the atoms are deflected because of the interaction between the dipole moment of the atoms and the magnetic field.

The net result is a selective measurement in which A_1, A_2 have definite values.

It follows that the operators representing compatible physical properties are communtative,

$$A_1 A_2 = A_2 A_1. \tag{12.2}$$

To prove this, remember how operators are constructed in terms of measurement symbols,

$$A_1 = \sum_{a_1'} a_1' |a_1'|, \quad A_2 = \sum_{a_2'} a_2' |a_2'|. \tag{12.3}$$

Then

$$A_1 A_2 = \sum_{a_1'} a_1' |a_1'| \sum_{a_2'} a_2' |a_2'| = \sum_{a_1' a_2'} a_1' a_2' |a_1' a_2'|, \tag{12.4a}$$

$$A_2 A_1 = \sum_{a_2'} a_2' |a_2'| \sum_{a_1'} a_1' |a_1'| = \sum_{a_2' a_1'} a_2' a_1' |a_2' a_1'|, \tag{12.4b}$$

so because the order of the labels in the measurement symbol do not matter, that is, the order of selection makes no difference, it follows that

$$A_1 A_2 - A_2 A_1 \equiv [A_1, A_2] = 0, \tag{12.5}$$

that is, the *commutator* of two compatible operators vanishes.

By measuring property 1 and property 2, one selects a state which has a definite value of both properties. Instead of describing the system by one number, we have a description in terms of 2, or more, numbers. Everything is really the same.

For a single physical property, recall that we had

$$|a'||a''| = \delta(a', a'')|a'|, \tag{12.6a}$$

and

$$\sum_{a'} |a'| = 1, \tag{12.6b}$$

where the latter represents a totally nonselective measurement, or no measurement at all, where all systems are selected. What's involved here is the question of identity ($a' = a''$), distinction ($a' \neq a''$), and completeness ($\sum_{a'} |a'| = 1$).

Now consider two compatible properties, and two selections. Then the counterpart of Eq. (12.6a) is

$$\begin{aligned}|a_1' a_2'||a_1'' a_2''| &= |a_1'||a_2'||a_1''||a_2''| = |a_1'||a_1''||a_2'||a_2''| \\ &= \delta(a_1', a_1'')|a_1'|\delta(a_2', a_2'')|a_2'| = \delta(a_1', a_1'')\delta(a_2', a_2'')|a_1' a_2'|, \quad (12.7)\end{aligned}$$

where we might write

$$\delta(a_1', a_1'')\delta(a_2', a_2'') \equiv \delta(a_1' a_2', a_1'' a_2'') = \begin{cases} 1 & \text{if}\, a_1' = a_1''\, \textit{and}\, a_2' = a_2'' \\ 0 & \text{otherwise, if}\, \{a_1', a_2'\} \neq \{a_1'', a_2''\}. \end{cases} \quad (12.8)$$

The product of measurement symbols is zero unless the pair of values is identical. That is, the second measurement annihilates the first unless it is an exact replication of the first. *Identity* means all (both) properties are the same, whicle *distinction* means that at least one property is different.

Compactly, we can still write

$$|a'||a''| = \delta(a', a'')|a'|, \quad (12.9)$$

where $a' = a_1' a_2'$, the pair of numbers.

What about completeness?

$$\sum_{a_1' a_2'} |a_1' a_2'| = \sum_{a_1' a_2'} |a_1'||a_2'| = \sum_{a_1'} |a_1'| \sum_{a_2'} |a_2'| = 1 \cdot 1 = 1, \quad (12.10)$$

since all systems have some possible value of A_1 and A_2.

The only difference now is that a state is specified not by one number, but by two.

In general, there might be c compatible properties specifying a state:

$$A_1,\, A_2,\, A_3,\, \ldots, A_c. \quad (12.11)$$

If these are compatible, they must all commute with each other,

$$[A_i, A_j] = 0. \quad (12.12)$$

We have a lot of freedom in choosing compatible properties. Ordinarily, there are only a finite number of independent compatible operators. Any other property is either

1. a function of the other properties, or

2. is *incompatible* with the other properties.

If B is incompatible with A this means the commutator of these two operators is not zero,

$$[B, A] \neq 0. \tag{12.13}$$

To specify a state, we measure all of its compatible properties,

$$|a_1'||a_2'|\cdots|a_c'| = |a_1' a_2' \ldots a_c'|, \tag{12.14}$$

where the labels are the values of a complete set of compatible properties. If we try to measure more, we will disturb the values found by previous measurements. For shorthand we will write

$$|a'| = |a_1' a_2' \ldots a_c'|. \tag{12.15}$$

A selective measurement picks out atoms in a definite *state*: $|a'|$. A state is determined when all compatible properties are specified.

Now recall we also talked about selective measurements in which a state changes, which we decomposed into right and left vectors,

$$|a'a''| = |a'\rangle\langle a''|; \tag{12.16}$$

now the a' is to be regarded as a set of numbers.

We could have begun with a different physical property B_1 (say μ_x not μ_z), select a second property compatible with it, and so on. In this way we get a second complete set of physical properties,

$$B_1,\, B_2,\, ,\ldots, B_c, \quad [B_i, B_j] = 0. \tag{12.17}$$

A selective measurement in the B description is given by

$$|b_1' b_2' \cdots b_c'| \equiv |b'|, \tag{12.18}$$

and again,

$$|b'b''| = |b'\rangle\langle b''| \tag{12.19}$$

represents a measurement in which (reading from left to right) $b' = \{b_1', \ldots, b_c'\}$ is selected, but $b'' = \{b_1'', \ldots, b_c''\}$ is emitted.

The inner product of a left and right vector, belonging to the A and B descriptions respectively, is the probability amplitude (or transformation function),

$$\langle a'|b'\rangle, \tag{12.20}$$

where again the a''s and b''s are sets of numbers specifying the two states. The probability of measuring $a' = \{a_1', a_2', \ldots, a_c'\}$ if the state was previously selected to have $b' = \{b_1', b_2', \ldots, b_c'\}$ (or vice versa) is obtained by taking the absolute square of the probability amplitude,

$$p(a', b') = |\langle a'|b'\rangle|^2. \tag{12.21}$$

All that's involved now is a collection of numbers, rather than a single number.

Now let's express this in terms of eigenvectors. For a single property

$$(A - a')|a'\rangle = 0, \tag{12.22}$$

where $|a'\rangle$ is the eigenvector and a' is the eigenvalue. For c compatible properties, we have a set of eigenvalue equations:

$$(A_1 - a'_1)|a'\rangle = 0, \tag{12.23a}$$
$$(A_2 - a'_2)|a'\rangle = 0, \tag{12.23b}$$
$$\ldots, \tag{12.23c}$$
$$(A_c - a'_c)|a'\rangle = 0, \tag{12.23d}$$

which says $|a'\rangle$ is a simultaneous eigenvector of A_1, $A_2, \ldots$, A_c. Now pick out the ith and jth properties; the eigenstate satisfies

$$(A_i - a'_i)|a'\rangle = 0, \quad (A_j - a'_j)|a'\rangle = 0, \tag{12.24}$$

so we conclude

$$(A_j - a'_j)(A_i - a'_i)|a'\rangle = 0, \quad (A_i - a'_i)(A_j - a'_j)|a'\rangle = 0, \tag{12.25}$$

or if we subtract these two equations,

$$[A_i - a'_i, A_j - a'_j]|a'\rangle = 0. \tag{12.26}$$

Now the commutator obeys the properties of linearity:

$$[X + Y, Z] = [X, Z] + [Y, Z], \tag{12.27a}$$
$$[X, Y + Z] = [X, Y] + [X, Z]. \tag{12.27b}$$

Therefore we conclude

$$[A_i - a'_i, A_j - a'_j] = [A_i, A_j] - a'_i[1, A_j] - a'_j[A_i, 1] + a'_i a'_j[1, 1]. \tag{12.28}$$

But unity commutes with everything, so the above is

$$[A_i - a'_i, A_j - a'_j] = [A_i, A_j] = 0, \tag{12.29}$$

because the properties are compatible. Further, if

$$[A_i, A_j]|a'\rangle = 0 \tag{12.30}$$

holds for a complete set of eigenstates, then by summing over all the states,

$$\sum_{a'}[A_i, A_j]|a'| = [A_i, A_j]\sum_{a'}|a'| = [A_i, A_j] = 0. \tag{12.31}$$

In general, we cannot have a vector being an eigenstate of two operators unless the operators commute. The theory itself must tell us what operators, properties are compatible. The theory itself must tell us what the properties are.