Chapter 9

Summary: Construction of Quantum Kinematics

In this course, we have inferred the construction of quantum mechanics from the bottom up, starting from an analysis of the results of the Stern-Gerlach experiment, in which only discrete outcomes result. From this we described general measurements in terms of measurement symbols referring to a property A, which takes on n discrete values

$$A: \quad \{a'\} = \{a_1, a_2, a_3, \dots, a_n\}. \tag{9.1}$$

The measurement symbol that represents a selective measurement in which only systems or "atoms" with $A = a'$ are selected, all of which selected atoms are emitted from the measurement apparatus in the state $A = a''$ is

$$|a'a''|. \tag{9.2}$$

We subsequently realized that it was useful to factor this symbol into the (outer) product of two vectors,

$$|a'a''| = |a'\rangle\langle a''|, \tag{9.3}$$

where now we say that the state of the system is described either in terms of a right vector $|a'\rangle$, or the corresponding left vector $\langle a'|$.

Addition of measurement symbols corresponds to a less selective measurment:

$$|a'a'| + |a''a''| \tag{9.4}$$

represents a measurement in which a' is selected (and emitted) *or* a'' is selected (and emitted), *without discrimination.* In particular, if all possible outcomes are selected without discriminiation,

$$\sum_{a'} |a'a'| = 1, \tag{9.5}$$

where the summation sign represent the sum over all possible values A can assume, because the net effect is to allow all states to pass through the measuring apparatus. Written in terms of vectors, this is a statement that the a vectors are complete:

$$\sum_{a'} |a'\rangle\langle a'| = 1. \tag{9.6}$$

The statement that $1|a'\rangle = |a'\rangle$ then inplies that these vectors form an orthonormal set:

$$\langle a'|a''\rangle = \delta(a', a'') = \begin{cases} 1, & a' = a'', \\ 0, & a' \neq a''. \end{cases} \tag{9.7}$$

The physical property A can be represented itself by a symbol or operator,

$$A = \sum_{a'} a'|a'a'| = \sum_{a'} a'|a'\rangle\langle a'|. \tag{9.8}$$

This in turn implies the following eigenvalue equation:

$$A|a'\rangle = a'|a'\rangle, \tag{9.9}$$

so that $|a'\rangle$ is an eigenvector of the operator A, with eigenvalue a'.

The adjoint maps left vectors into right vectors, that is, reverses the convention of reading the symbols:

$$|a'a''|^\dagger = |a''a'|, \quad |a'\rangle^\dagger = \langle a'|, \quad \langle a'|^\dagger = |a'\rangle. \tag{9.10}$$

Now, because a physical property must have only real values, it follows that it must be self-adjoint or Hermitian:

$$A^\dagger = \sum a'|a'a'| = A, \tag{9.11}$$

and that the adjoint of the eigenvalue equation (9.9) is

$$\langle a'|A = \langle a'|a', \tag{9.12}$$

which implies the orthogonality equation (9.7).

Now consider an arbitrary operator, not necessary a Hermitian one. It can be expanded in terms of measurement symbols,

$$X = \sum_{a'a''} |a'\rangle\langle a'|X|a''\rangle\langle a''| = \sum_{a'a''} \langle a'|X|a''\rangle|a'a''|, \tag{9.13}$$

where $\langle a'|X|a''\rangle$ is the a'-a'' matrix element of X. That is, X is represented by an $n \times n$ matrix, with the element in the a'th row and a''th column being $\langle a'|X|a''\rangle$. If we take the adjoint of this,

$$X^\dagger = \sum_{a'a''} \langle a'|X|a''\rangle^* |a''a'| = \sum_{a'a''} \langle a''|X^\dagger|a'\rangle|a''a'|, \tag{9.14}$$

so

$$\langle a'|X|a''\rangle = \langle a''|X|a'\rangle^*, \tag{9.15}$$

so the matrix of $X^\dagger$ is the transposed, complex conjugate of that for X. Here are the properties of the adjoint:

$$(X+Y)^\dagger = X^\dagger + Y^\dagger, \tag{9.16a}$$

$$(\lambda X)^\dagger = \lambda^* X^\dagger, \tag{9.16b}$$

$$(XY)^\dagger = Y^\dagger X^\dagger. \tag{9.16c}$$

Instead of using A to describe the system, we could use another property B. (We assume that specifying either A or B fully describes the system: knowing A precludes definite knowledge of property B, like σ_z and σ_y.) We could then equally well introduce eigenstates of B:

$$B|b'\rangle = b'|b'\rangle, \tag{9.17}$$

where b' is one of a set of n eigenvalues, and the eigenvectors are both complete and orthonormal:

$$\sum_{b'} |b'b'| = \sum_{b'} |b'\rangle\langle b'| = 1, \tag{9.18a}$$

$$\langle b'|b''\rangle = \delta(b', b''). \tag{9.18b}$$

We can then pass from the b description to the a description,

$$|b'\rangle = \sum_{a'} |a'\rangle\langle a'|b'\rangle, \quad \langle b'| = \sum_{a'} \langle b'|a'\rangle\langle a'|, \tag{9.19}$$

where the set of numbers $\langle a'|b'\rangle$ are called the transformation function. (The second equation here is the adjoint of the first.) They are like the direction cosines between two different set of coordinate axes describing ordinary three-dimensional vectors. The a'-b' and b'-a' transformation functions are related by complex conjugation:

$$\langle a'|b'\rangle = \langle b'|a'\rangle^*. \tag{9.20}$$

But the transformation function elements are also probability amplitudes: the probability of measuring $B = b'$ in a state prepared so that $A = a'$, or the probability of measuring $A = a'$ in a state prepared so that $B = b'$, is

$$p(a', b') = p(b', a') = |\langle a'|b'\rangle|^2. \tag{9.21}$$

Not only are these probabilities positive, but they possess the property

$$\sum_{a'} p(a', b') = \sum_{a'} \langle b'|a'\rangle\langle a'|b'\rangle = \langle b'|b'\rangle = 1, \tag{9.22}$$

that is, the likelihood of *some* outcome is unity.

In general, an arbitrary state is described by a unit vector $|?\rangle$,

$$\langle ?|?\rangle = 1. \tag{9.23}$$

In the a description, the wavefunction of this state is

$$\psi_?(a') = \langle a'|?\rangle. \tag{9.24}$$

This is the probability amplitude of finding the system, prepared in the ? state, with the value $A = a'$:

$$p(a', ?) = |\langle a'|?\rangle|^2 = |\psi_?(a')|^2. \tag{9.25}$$

Again, probability is conserved:

$$\sum_{a'} p(a', ?) = \sum_{a'} \langle ?|a'\rangle\langle a'|?\rangle = 1. \tag{9.26}$$

Changes of the coordinate system, for example, under rotations, are described by unitary transformations, in terms of a unitary operator,

$$U^\dagger = U^{-1}. \tag{9.27}$$

Under unitary transformation, state vectors transform thusly:

$$\langle a'| \to \overline{\langle a'|} = \langle a'|U, \quad |a'\rangle \to \overline{|a'\rangle} = U^\dagger|a'\rangle. \tag{9.28}$$

Operators transform this way:

$$X \to \overline{X} = U^\dagger X U, \tag{9.29}$$

so inner products and matrix elements are unchanged:

$$\langle a'|a''\rangle \to \overline{\langle a'|a''\rangle} = \langle a'|UU^\dagger|a''\rangle = \langle a'|a''\rangle, \tag{9.30a}$$

$$\langle a'|X|a''\rangle \to \overline{\langle a'|}\overline{X}\overline{|a'\rangle} = \langle a'|UU^\dagger XUU^\dagger|a''\rangle = \langle a'|X|a''\rangle. \tag{9.30b}$$

So are eigenvalues:

$$A|a'\rangle = a'|a'\rangle \to \overline{A}\overline{|a'\rangle} = U^\dagger AUU^\dagger|a'\rangle = U^\dagger a'|a'\rangle = a'\overline{|a'\rangle}. \tag{9.31}$$

For example, for spin-1/2, consider a rotation carrying us from the x, y, z coordinate system, to a new coordinate system x', y', z', where

$$\sigma_{z'} = \sigma_x \sin\theta\cos\phi + \sigma_y \cos\theta\sin\phi + \sigma_z\cos\theta. \tag{9.32}$$

This can be implemented by a unitary transformation

$$\sigma'_z = U^\dagger\sigma_z U, \tag{9.33}$$

where

$$U = e^{i\frac{\theta}{2}\sigma_y}e^{i\frac{\phi}{2}\sigma_z}, \tag{9.34}$$

by use of the properties of the σ matrices or operators,

$$\sigma_x\sigma_y = -\sigma_x\sigma_y = i\sigma_z, \quad \sigma_x^2 = 1, \tag{9.35}$$

and so on by cyclic permutations. Thus, for an arbitrary direction, the possible values of $\sigma_{z'}$ are ± 1.