Second Examination
Physics 3803
Introduction to Quantum Mechanics I

April 11, 2012

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions. Remember this is a closed book, closed notes, exam. *Good luck!*

1. If U is a unitary operator, $U^\dagger = U^{-1}$, and λ is an eigenvalue of U, show that
$$|\lambda| = 1.$$
[Hint: U preserves the lengths of vectors.]

2. (a) Find, by solving the eigenvalue equation,
$$\sigma_y|\sigma_y'\rangle = \sigma_y'|\sigma_y'\rangle,$$
the eigenvalues and normalized eigenvectors of
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$
(b) Now compute the new matrix with elements
$$\langle\sigma_y'|\sigma_y|\sigma_y''\rangle.$$
(c) Similarly, compute the new matrices
$$\langle\sigma_y'|\sigma_x|\sigma_y''\rangle, \quad \langle\sigma_y'|\sigma_z|\sigma_y''\rangle.$$

(d) Show that the product of these last two matrices is $-i$ times the matrix found in part 2b.

3. The general angular momentum commutation relation is

$$[J_i, J_j] \equiv J_i J_j - J_j J_i = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} J_k,$$

in terms of the totally antisymmetric object ϵ_{ijk}, or

$$[J_x, J_y] = i\hbar J_z,$$

and so on by cyclic permutation. Using this compute

$$\begin{aligned} e^{-i\phi J_z/\hbar} J_x e^{i\phi J_z/\hbar} &= J_x(\phi), \\ e^{-i\phi J_z/\hbar} J_y e^{i\phi J_z/\hbar} &= J_y(\phi), \\ e^{-i\phi J_z/\hbar} J_z e^{i\phi J_z/\hbar} &= J_z(\phi). \end{aligned}$$

You should obtsin, for example,

$$J_x(\phi) = J_x \cos\phi + J_y \sin\phi.$$

[Hint: Compute

$$\frac{d}{d\phi} J_x(\phi), \quad \frac{d}{d\phi} J_y(\phi),$$

taking care not to interchange non-commuting factors, using $\frac{d}{dx} e^x = e^x$. Integrate the resulting differential equations. Note that the result just expresses how the components of a vector change under a coordinate system rotation about the z axis, and is an immediate generalization of what we have done repeatedly for spin 1/2.]