Introduction to Quantum Mechanics I
9th Homework Assignment
Due: Friday, April 20, 2012

April 13, 2012

1. Given the matrix elements $\langle a'|X|a''\rangle$, what are the $\langle b'|X|b''\rangle$'s?. Use this relation to prove
$$\sum_{b'}\langle b'|X|b'\rangle = \sum_{a'}\langle a'|X|a'\rangle.$$

2. Show that
$$\sum_{a'a''}|a'a''|X|a''a'| = 1\,\mathrm{tr}\,X.$$
For $n = 2$, convert this statement to
$$\frac{1}{2}(X + \boldsymbol{\sigma}\cdot X\boldsymbol{\sigma}) = 1\,\mathrm{tr}\,X,$$
and check this last for $X = 1$, $X = \sigma_k$.

3. (a) Prove $\mathrm{tr}\,X^\dagger X \geq 0$; when is it zero?

   (b) Show that
$$\mathrm{tr}\,(|a'a''|^\dagger|a'''a^{iv}|) = \delta(a', a''')\delta(a''a^{iv}).$$

4. Demonstrate that
$$|\mathrm{tr}\,X^\dagger Y|^2 \leq (\mathrm{tr}\,X^\dagger X)(\mathrm{tr}\,Y^\dagger Y).$$
[Hint: Consider $X + \lambda Y$ for arbitrary λ and apply Eq. (3a).] Check this result is true for $X = |a'|$, $Y = |b'|$.

5. What follows similarly from $\langle 1|1\rangle \geq 0$ and the consideration of $|1\rangle + \lambda|2\rangle$? Can you think of a physical example?

6. (a) Prove $\operatorname{tr} A = \operatorname{tr} U^{-1}AU$.

   (b) If A is a $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, prove that

$$\operatorname{tr} A = \sum_{j=1}^{n} \lambda_j = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

   [Hint: Recall problem 8 of assignment 8.]