# Introduction to Quantum Mechanics I
# 8th Homework Assignment
# Due: Monday, April 9, 2012

April 7, 2012

1. Consider the $3 \times 3$ matrices

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Are these Hermitian? Compute $s_k^2$ for $k = 1, 2, 3$. Verify

$$(1 - s_k^2) s_k = 0.$$

   Compute $\sum_{k=1}^{3} s_k^2$. Can you suggest a physical interpretation for these matrices?

2. Show, by example, that

$$s_k^2 s_l + s_l s_k^2 = s_l, \quad k \neq l,$$
$$s_k s_l s_m + s_m s_l s_k = 0, \quad k \neq l \neq m.$$

   Compute $\mathbf{s} \times \mathbf{s}$, where, for example,

$$(\mathbf{s} \times \mathbf{s})_3 = s_1 s_2 - s_2 s_1.$$

3. Solve the eigenvalue problem

$$s_3 |s_3'\rangle = s_3' |s_3'\rangle,$$

where $|s_3'\rangle$ is represented by the column vector (wavefunction),

$$\begin{pmatrix} \psi(1) \\ \psi(2) \\ \psi(3) \end{pmatrix}.$$

That is, find the eigenvalues $s_3'$ and the corresponding orthonormal wavefunctions.

4. Use matrices and wavefunctions to determine

$$(s_1 + is_2)|s_3'\rangle, \quad (s_1 - is_2)|s_3'\rangle,$$

(Use the possibility of additional minus signs in wavefunctions to get positive numerical coefficients in the answers.)

5. In terms of the eigenfunctions determined in Problem 3, compute the matrix elements

$$\langle s_3'|s_k|s_3''\rangle,$$

and hence the *new* matrices representing the $s$'s. Do these matrices have the same properties as the original matrices found in Problems 1 and 2?

6. From $\langle 1|UU^\dagger|1\rangle = \langle 1|1\rangle$ for all vectors $|1\rangle$, show that

$$UU^\dagger = 1, \quad U^\dagger = U^{-1}$$

is not only sufficient (it works) but necessary (nothing else works).

7. Show that a Hermitian operator $A$ has only real eigenvalues. [Hint: If $A$ is Hermitian, the eigenvalue equation

$$A|a'\rangle = a'|a'\rangle$$

implies

$$\langle a'|A = \langle a'|a'^*.$$

What can you then conclude from considering

$$\langle a'|A|a''\rangle?]$$

8. Suppose $A$ is a Hermitian matrix. Construct another matrix $U$ by placing in each column the normalized eigenvectors of $A$:

$$U_{ij} = \psi_j(i),$$

where $A\psi_j = \lambda_j \psi_j$, $\lambda_j$ being the eigenvalue, and $\psi_j(i)$ is the $i$th component of $\psi_j$. Show that

(a) $U$ is unitary.

(b) $(AU)_{ij} = \lambda_j U_{ij}$ (no summation on repeated indices).

(c) $U^{-1}AU$ is a diagonal matrix. What are the diagonal matrix elements?