Introduction to Quantum Mechanics I
7th Homework Assignment
Due: Wednesday, March 28, 2012

March 26, 2012

1. Show that the wavefunctions for spin 1/2 derived in class [Eq. (7.36)] give the same probabilities in an arbitrary coordinate system

$$p(m, m') = |\psi_m(\theta, \phi)^\dagger \psi_{m'}(\theta', \phi')|^2$$

as the wavefunctions derived earlier [Eq. (4.35)]. Can you deduce any general principle from this?

2. As stated in problem 4 of set #5, the most general unitary transformation representing a rotation of the coordinate system through the Euler angles ψ, θ, ϕ is

$$U = e^{i\frac{\psi}{2}\sigma_z} e^{i\frac{\theta}{2}\sigma_y} e^{i\frac{\phi}{2}\sigma_z}.$$

Using the method given in class, work out the corresponding wavefunctions

$$\psi_{\sigma''_{z'}}(\sigma'_z) = \psi_{\pm z'}(\pm z) = \langle \pm z | \pm z' \rangle,$$

where the two sets of $\pm$ signs are independent. How do these differ from those found in class? Will these lead to different probabilities?

3. The proton and neutron can be usefully thought of as two states of a single particle, the *nucleon*, distinguished by the value of electric charge. (This is useful because nuclear forces are charge independent.)

In units of e, the magnitude of the charge on the electron, their charges are

$$q = \begin{cases} \text{proton:} & 1 \\ \text{neutron:} & 0 \end{cases}.$$

Define

$$\tau_3 = 2q - 1 = \begin{cases} \text{proton:} & +1 \\ \text{neutron:} & -1 \end{cases}.$$

This is a two-level system, precisely analogous to a spin-1/2 system. The new property involved here is called *isospin*. The nucleon is a isospin-1/2 (symbolically $T = 1/2$) particle; the proton (p) is the isospin-up state ($T_3 = +1/2$) while the neutron (n) is the isospin-down state ($T_3 = -1/2$). In analogy with the spin-1/2 objects σ_1, σ_2, σ_3, introduce, by way of measurement symbols, in addition to τ_3, new objects τ_1 and τ_2, which together form an algebraically identical (physically quite different) set of elements to the $\boldsymbol{\sigma}$s. Verify that

$$\frac{1}{2}(\tau_1 + i\tau_2)|n\rangle = |p\rangle, \quad \frac{1}{2}(\tau_1 - i\tau_2)|p\rangle = |n\rangle.$$

and compute

$$\frac{1}{2}(\tau_1 + i\tau_2)|p\rangle, \quad \frac{1}{2}(\tau_1 - i\tau_2)|n\rangle.$$

Work out the similar left-vector statements.

4. Using the σ-*matrices* work out explicitly as matrices

$$U = e^{i\frac{\theta}{2}\sigma_y} e^{i\frac{\phi}{2}\sigma_z}, \quad U^{-1} = e^{-i\frac{\phi}{2}\sigma_z} e^{-i\frac{\theta}{2}\sigma_y},$$

and verify, as matrices,

$$UU^{-1} = U^{-1}U = 1.$$

Then compute the new matrix

$$\sigma_{z'} = U^{-1}\sigma_z U.$$

Verify that the wavefunction $\psi_{\pm z'}$ represents a state in which $\sigma_{z'}$ has a definite value by computing

$$\sigma_{z'}\psi_{\pm z'}.$$