Introduction to Quantum Mechanics I
6th Homework Assignment
Due: Wednesday, March 14, 2012

March 4, 2012

1. Find in general the probabilities of outcomes of two successive Stern-Gerlach experiments on spin-1/2 atoms, $p(\sigma'_z, \sigma''_{z'})$, from

$$|\sigma''_{z'}||\sigma'_z||\sigma''_{z'}| = p(\sigma'_z, \sigma''_{z'})|\sigma''_{z'}|,$$

following the same procedure given in class for $\sigma'_z = \sigma''_{z'} = +1$. The successive experiments correspond to first selecting $\sigma_{z'} = \sigma''_{z'}$ and then selecting $\sigma_z = \sigma'_z$, that is, selecting spins in the z' and z directions, respectively.

2. Compute $p(\sigma'_z, \sigma''_{z'})$ from the above formula, but in a different way, making use of the unitary transformation

$$\sigma_{z'} = U^{-1}\sigma_z U, \quad U = e^{i\frac{\theta}{2}\sigma_y} e^{i\frac{\phi}{2}\sigma_z}.$$

[Hint: as an intermediate step, obtain

$$\frac{1+\sigma''_{z'}\sigma_z}{2} U \frac{1+\sigma'_z\sigma_z}{2} U^{-1} \frac{1+\sigma''_{z'}\sigma_z}{2} = p(\sigma'_z, \sigma''_{z'}) \frac{1+\sigma''_{z'}\sigma_z}{2},$$

and then use

$$e^{i\frac{\theta}{2}\sigma_y} = \cos\frac{\theta}{2} + i\sigma_y \sin\frac{\theta}{2}.]$$

3. The weighted average of A values,

$$\sum_{a'} a' p(a', b') \equiv \langle A \rangle_{b'},$$

is called the expectation value of A in the state b'. Show that

$$|b'|A|b'| = \langle A \rangle_{b'} |b'|,$$

and then derive the relation

$$\langle b'|A|b' \rangle = \langle A \rangle_{b'}.$$

4. Show that

$$\sum_{k=1}^{3} \sigma_k \boldsymbol{\sigma} \sigma_k + \boldsymbol{\sigma} = 0.$$

5. The vector product is defined by

$$(\mathbf{A} \times \mathbf{B})_3 = A_1 B_2 - A_2 B_1,$$

and similarly (cyclically) for the 1 and 2 components. If $A_{1,2,3}$ are elements of a non-commutative algebra, the components of $\mathbf{A}$ do not commute, and $\mathbf{A} \times \mathbf{A} \neq 0$ in general. Show that

$$\frac{1}{2}\boldsymbol{\sigma} \times \frac{1}{2}\boldsymbol{\sigma} = i\frac{1}{2}\boldsymbol{\sigma}.$$

This is equivalent to the angular momentum statement

$$\mathbf{J} \times \mathbf{J} = i\hbar \mathbf{J}.$$