Introduction to Quantum Mechanics I
5th Homework Assignment
Due: Monday, March 5, 2012

March 2, 2012

1. Show that
$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \quad \text{and} \quad \sigma_x \sigma_y = i\sigma_z$$
imply the remaining multiplication properties of the σ's.

2. Consider a rotation of the coordinate system about the z axis through an angle ϕ as shown in Fig. 1. The new components of σ are
$$\begin{aligned} \sigma_{x'} &= \sigma_x \cos\phi + \sigma_y \sin\phi, \\ \sigma_{y'} &= -\sigma_x \sin\phi + \sigma_y \cos\phi, \\ \sigma_{z'} &= \sigma_z. \end{aligned}$$
Verify directly that the new σ components have the same algebraic properties as the old σ components:
$$\sigma_{x'}^2 = \sigma_{y'}^2 = 1, \quad \sigma_{x'}\sigma_{y'} = i\sigma_{z'}.$$

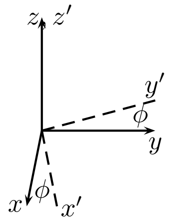

Figure 1: Rotation of a Cartesian coordinate system through an angle ϕ about the z axis.

3. This problem supplies a glimpse into the future. Define

$$\sigma_4 = i1,$$

where 1 is the unit symbol, and with $k = 1, 2, 3,$ or 4 let

$$\sigma_{k'} = U \sigma_k U$$

(the fact that the first factor on the right of the equals sign is not U^{-1} is *not* a misprint), with

$$U = e^{i\frac{\phi}{2}\sigma_3}$$

verify that

$$\begin{aligned}
\sigma_{1'} &= \sigma_1, \\
\sigma_{2'} &= \sigma_2, \\
\sigma_{3'} &= \sigma_3 \cos\phi + \sigma_4 \sin\phi, \\
\sigma_{4'} &= -\sigma_3 \sin\phi + \sigma_4 \cos\phi.
\end{aligned}$$

What happens when

$$U = e^{i\frac{\phi}{2}\sigma_1}?$$

These transformations (which are not unitary) represent four-dimensional rotations related to the Lorentz transformations of relativity.

4. A general rotation of a coordinate system is specified by not two angles, but by three, the Eulerian angles ψ, θ, ϕ. The corresponding unitary transformation is given by, for spin 1/2,

$$U = e^{\frac{i}{2}\psi\sigma_z} e^{\frac{i}{2}\theta\sigma_y} e^{\frac{i}{2}\phi\sigma_z}.$$

Verify that, with this form for U,

$$\sigma_{z'} = U^{-1}\sigma_z U = \sigma_x \sin\theta\cos\phi + \sigma_y \sin\theta \sin\phi + \sigma_z \cos\theta,$$

as shown in class, and compute

$$\begin{aligned}
\sigma_{x'} &= U^{-1}\sigma_x U, \\
\sigma_{y'} &= U^{-1}\sigma_y U.
\end{aligned}$$