Introduction to Quantum Mechanics I
3rd Homework Assignment
Due: Monday, February 13, 2012

February 6, 2012

1. The purpose of this heuristic problem is to derive the probabilities for the various outcomes for two successive Stern-Gerlach experiments performed on atoms of spin 1. The method is a generalization of that used for spin 1/2 in class. Suppose the first measurement deflects the beam in the z' direction, by selecting $J_{z'} = m'\hbar$. (Other values of $J_{z'}$ are rejected.) Once we measure $J_{z'}$ we cannot know the precession angle in the x', y' plane, that is

$$\langle J_{x'} \rangle_{m'} = \langle J_{y'} \rangle_{m'} = 0, \tag{1}$$

where the m' subscript means that the average is taken on those atoms which have $J_{z'} = m'\hbar$. However, show that

$$\langle J_{x'}^2 \rangle_{m'} = \langle J_{y'}^2 \rangle_{m'} = \langle \frac{J_{x'}^2 + J_{y'}^2}{2} \rangle_{m'} = \frac{1}{2}\left[j(j+1) - m'^2 \right] \hbar^2. \tag{2}$$

On the other hand, since $J_{x'}$ or $J_{y'}$ is as equally likely to be positive as negative,

$$\langle J_{x'} J_{y'} \rangle_{m'} = 0. \tag{3}$$

Now send this selected beam through a second Stern-Gerlach apparatus. The second magnetic field is inclined with respect to the first by polar angle θ, ϕ, as shown in Fig. 1. Show that

$$J_z = J_{x'} \sin\theta \cos\phi + J_{y'} \sin\theta \sin\phi + J_{z'} \cos\theta, \tag{4}$$

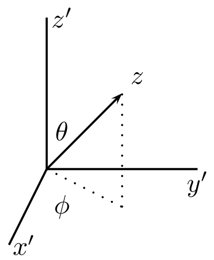

Figure 1: The first Stern-Gerlach apparatus has a magnetic field oriented in the z' direction. The second Stern-Gerlach apparatus has a magnetic field in the z direction. This figure shows the polar angles determining the direction of z in the coordinate system established by the first apparatus.

and consequently that the average value of J_z measured by the second apparatus, when the beam of atoms coming into it is known to have $J_{z'} = m'\hbar$ is

$$\langle J_{z'} \rangle_{m'} = m'\hbar \cos\theta. \tag{5}$$

Using what we mean by probabilities, show that this is equivalent to the equation

$$\sum_{m=-j}^{j} m\, p(m, m') = m' \cos\theta, \tag{6}$$

where $p(m, m')$ is the probability of measuring $J_z = m\hbar$ in the second apparatus, given that the first apparatus determined $J_{z'} = m'\hbar$. The fact that the probability of having some outome is unity supplies another equation,

$$\sum_{m=-j}^{j} p(m, m') = 1. \tag{7}$$

In all there are $(2j+1)^2$ probabilities $p(m, m')$, but we have here only $2(2j+1)$ equations. This is sufficient for $j = 1/2$, and these are just the equations we solved in class. For $j = 1$, however, there are 9 $p(m, m')$'s, and only 6 equations so far. We need three more equations. Derive the following set of equations:

$$\langle J_z^2 \rangle_{m'} = (m'\hbar)^2 \cos^2\theta + \frac{\hbar^2}{2} \left[j(j+1) - m'^2 \right] \sin^2\theta. \tag{8}$$

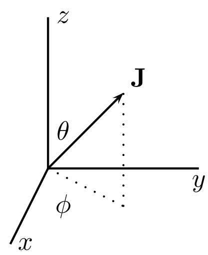

Figure 2: **J** in an arbitrary coordinate system.

From this obtain

$$\sum_{m=-j}^{j} m^2 p(m, m') = \frac{1}{2} j(j+1) \sin^2\theta + m'^2 \frac{1}{2}(3\cos^2\theta - 1). \quad (9)$$

Show that this equation is satisfied for $j = 1/2$, and use it, together with the other probability equations, to determine all 9 $p(m.m')$'s for $j = 1$.

2. Using the result of Problem 1, show that

$$p(0,0) = |\psi_0(\theta, \phi)^\dagger \psi_0(\theta', \phi')|^2, \quad (10)$$

where $\psi_0(\theta, \phi)$ is a three-component (column) vector or wavefunction for the spin-1 system. What is this wavefunction?

3. In measuring J_z, all directions in the x, y plane are equivalent. One way of saying this is in terms of the azimuthal angle ϕ (see Fig. 2):

$$\langle (J_x \cos\phi + J_y \sin\phi)^2 \rangle \quad (11)$$

is independent of ϕ. From this what can you say about

$$\langle J_x^2 \rangle, \quad \langle J_y^2 \rangle, \quad \langle J_x J_y \rangle? \quad (12)$$