Introduction to Quantum Mechanics I
10th Homework Assignment
Due: Wednesday, May 2, 2012

April 30, 2012

1. Solve the equations of motion (10.56) directly, by differentiating the equations with respect to t and then using the same equations again. In this way obtain a second order differential equation for $\psi(+)$ and for $\psi(-)$. Solve these equations by using the initial conditions

$$\psi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In this way obtain

$$\psi_{+z}(+, t), \quad \psi_{+z}(-, t)$$

corresponding to Eqs. (10.74) and (10.76).

2. Show that if the Hamiltonian is invariant under a rotation about the z axis, so that

$$H = e^{-iJ_z\theta/\hbar} H e^{iJ_z\theta/\hbar},$$

J_z is a constant of the motion.

3. Pure states are characterized by state vectors, such as $|1\rangle$, or the corresponding measurement symbols $|1\rangle\langle 1|$. If the eigenvectors of a physical property A are $|a'\rangle$,

$$A|a'\rangle = a'|a'\rangle,$$

show that

$$\langle 1|A|1\rangle = \sum_{a'} a' p(a', 1),$$

where $p(a', 1)$ is the probability of finding the value $A = a'$ in the state $|1\rangle$. Thus, $\langle 1|A|1\rangle = \langle A\rangle_1$ is the mean value or the expectation value of A in the state $|1\rangle$. Show that

$$\langle A\rangle_1 = \langle 1|A|1\rangle = \operatorname{tr}\rho A,$$

where $\rho = |1\rangle\langle 1|$.

4. In general, quantum systems exist not in pure states, but *mixed* states, characterized by a linear combination of measurement symbols,

$$\rho = \sum_n p(n)|n\rangle\langle n|$$

where $\{|n\rangle\}$ is some complete set of vectors,

$$\sum_n |n\rangle\langle n| = 1, \quad \langle n|m\rangle = \delta_{n,m},$$

and $p(n)$ is the probability of finding the system in the state n,

$$\sum_n p(n) = 1, \quad 0 \leq p(n) \leq 1.$$

Show that the mean value of a physical property A in such a state is

$$\langle A\rangle = \operatorname{tr}\rho A.$$

The operator ρ is called the "density operator."

5. Now, suppose the dynamical variable $A(t)$ depends on time, whereas the state of the system is fixed ($|n\rangle$ and $p(n)$ do not depend on time). Using Heisenberg's equation of time evolution, show that

$$i\hbar\frac{d}{dt}\langle A(t)\rangle = \operatorname{tr}([H, \rho]A).$$

6. In general, the Hamiltonian of a system may depend explicitly upon time. We define a unitary time evolution operator by

$$\langle v', t| = \langle v', 0|U(t);$$

then use the Schrödinger equation to show that

$$i\hbar\frac{d}{dt}U(t) = U(t)H.$$

Show that the explicit time dependence of $U(t)$ is given by

$$i\hbar\frac{\partial}{\partial t}U(t) = HU(t).$$

These two expressions are no longer the same, because H is not a constant of the motion, and hence does not commute with the time evolution operator.