Introduction to Quantum Mechanics I
First Homework Assignment
Due: Monday, January 30, 2012

January 18, 2012

1. Use the Boltzmann distribution to calculate the mean speed of an atom in a gas at thermal equilibrium in three spatial dimensions,
$$\langle|\mathbf{v}|\rangle = \frac{\int d^3v\,|\mathbf{v}|\,e^{-mv^2/2kT}}{\int d^3v\,e^{-mv^2/2kT}}, \quad v^2 = \mathbf{v}\cdot\mathbf{v}. \tag{1}$$
Compare this with the rms speed, $\sqrt{\langle v^2\rangle}$, $v^2 = \mathbf{v}\cdot\mathbf{v}$.

2. Consider a gas, each atom of which has a magnetic dipole moment $\boldsymbol{\mu}$, in thermal equilibrium at temperature T. If no magnetic field is applied, the orientation of the dipoles will be random, and the average dipole moment will be zero,
$$\langle\boldsymbol{\mu}\rangle = 0. \tag{2}$$
If a magnetic field $\mathbf{H}$ is applied to the gas, the dipoles will tend to line up with the magnetic field. However, because of thermal motion, this alignment will not be perfect. The probability, in general, of finding a configuration of energy E in a system at thermal equilibrium at temperature T is proportional to the Boltzmann factor,
$$e^{-E/kT}. \tag{3}$$
The energy of interaction of a dipole with the magnetic field is $E = -\boldsymbol{\mu}\cdot\mathbf{H}$. Then, show that the average magnetic moment of the gas at temperature T is
$$\langle\boldsymbol{\mu}\rangle_T = \frac{\int d\Omega\,\boldsymbol{\mu}\,\exp\left(\boldsymbol{\mu}\cdot\mathbf{H}/kT\right)}{\int d\Omega\,\exp\left(\boldsymbol{\mu}\cdot\mathbf{H}/kT\right)}, \quad \boldsymbol{\mu}\cdot\mathbf{H} = \mu H\cos\theta, \tag{4}$$

where $d\Omega$ ($= d(\cos\theta)d\phi$ in polar coordinates) is the element of solid angle, the element of area on a sphere of unit radius; the polar angles θ, ϕ correspond to the orientation of the dipole moment. Noting that, by symmetry, $\langle\boldsymbol{\mu}\rangle_T$ lies in the direction of $\mathbf{H}$, evaluate $\langle\boldsymbol{\mu}\rangle_T$ for weak fields or high temperatures, where

$$\mu H \ll kT. \tag{5}$$

The result has the form

$$\langle\boldsymbol{\mu}\rangle_T = \alpha\frac{\mathbf{H}}{T}; \tag{6}$$

find α in terms if k, $\boldsymbol{\mu}$. This is the empirical law of Pierre Curie. [Hint: it is not necessary to carry out an explicit angular integration, if one notes that

$$\int d\Omega\,\mu_x^2 = \int d\Omega\,\mu_y^2 = \int d\Omega\,\mu_z^2.] \tag{7}$$

Typically, $\mu \sim 10^{-20}$ erg/gauss. Is the weak field condition satisfied at room temperature for a reasonable magnetic field?

3. The energy of a charge e moving with velocity $\mathbf{v}$ in an external electromagnetic field is

$$E = e\phi - \frac{e}{c}\mathbf{v}\cdot\mathbf{A}, \tag{8}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. The relation between $\mathbf{A}$ and the magnetic field $\mathbf{H}$ is

$$\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{9}$$

For a constant (homogenous in space) magnetic field $\mathbf{H}$, verify that

$$\mathbf{A} = \frac{1}{2}\mathbf{H}\times\mathbf{r} \tag{10}$$

is a possible vector potential. Then, by looking at the energy, identify the magnetic moment of the moving charge $\boldsymbol{\mu}$.