Introduction to Quantum Mechanics I
Final Examination

May 9, 2012

This is a closed-book, closed-notes examination. Please ask the instructor if you have any questions. Attempt all parts of the exam; do not spend too much time on any one part. **Good luck!**

1. An atom with spin 1/2 is in a uniform magnetic field $\mathbf{B}$, where the magnetic field points along the x axis. The Hamiltonian for the system is
$$H = -\gamma B \frac{\hbar}{2} \sigma_x,$$
where γ is a constant, the gyromagnetic ratio. Using the Schrödinger equation,
$$i\hbar \frac{\partial}{\partial t} \langle a', t| = \langle a', t| H,$$
solve for the probability amplitude
$$\langle +z, t| + z, t = 0\rangle,$$
and compute the probability that the atom, initially prepared (at time $t = 0$) in the state $| + z\rangle$, that is, with $\sigma'_z = +1$, remains in that state after a time t in the magnetic field,
$$p(+z, t; +z, 0).$$
What is the probability that the spin will flip,
$$p(-z, t; +z, 0)?$$

2. Angular momentum is represented by a vector operator with the properties
$$[J_x, J_y] = i\hbar J_z, \quad [J_y, J_z] = i\hbar J_x, \quad [J_z, J_x] = i\hbar J_y.$$
Here the commutator is defined by
$$[A, B] = AB - BA.$$

   (a) Define for arbitrary real θ
   $$\tilde{J}_x(\theta) = e^{-iJ_z\theta/\hbar} J_x e^{iJ_z\theta/\hbar},$$
   which represents a rotation about the z axis through an angle θ. Similarly define
   $$\tilde{J}_y(\theta) = e^{-iJ_z\theta/\hbar} J_y e^{iJ_z\theta/\hbar},$$
   Show that
   $$\frac{d}{d\theta}\tilde{J}_x(\theta) = \tilde{J}_y(\theta), \quad \frac{d}{d\theta}\tilde{J}_y(\theta) = -\tilde{J}_x(\theta),$$
   Show that the solution to these equations is, in part,
   $$\tilde{J}_x(\theta) = J_x \cos\theta + J_y \sin\theta.$$
   What is $\tilde{J}_y(\theta)$? Similarly compute $\tilde{J}_z(\theta)$.

   (b) Use the above results to evaluate the following product
   $$e^{i\pi J_y/\hbar} e^{-i\pi J_z/\hbar} e^{-i\pi J_y/\hbar} e^{i\pi J_z/\hbar}$$
   in two different ways to show
   $$e^{2i\pi J_z/\hbar} = e^{2i\pi J_y/\hbar}.$$
   [Hint:
   $$e^{i\pi J_y/\hbar} e^{-i\pi J_z/\hbar} e^{-i\pi J_y/\hbar} = e^{-i\tilde{J}_z(-\pi)/\hbar},$$
   where $\tilde{J}_z(-\pi)$ is the angular momentum operator J_z rotated through an angle $-\pi$ about the y axis.]

   (c) Similarly evaluate
   $$e^{i\pi J_y/\hbar} e^{i\pi J_z/\hbar} e^{-i\pi J_y/\hbar} e^{-i\pi J_z/\hbar}$$
   to show
   $$e^{-2i\pi J_z/\hbar} = e^{2i\pi J_y/\hbar}.$$

(d) Combine the results from the two previous parts to show

$$e^{4i\pi J_z/\hbar} = 1,$$

and conclude what are the possible values of J_z.

(e) Finally, evaluate in two ways

$$e^{i\frac{\pi}{2}J_y/\hbar}e^{-i\pi J_z/\hbar}e^{-i\frac{\pi}{2}J_y/\hbar}e^{i\pi J_z/\hbar}$$

to obtain

$$e^{i\pi J_x/\hbar}e^{i\pi J_z/\hbar} = e^{i\pi J_y/\hbar},$$

and show this is equivalent to

$$e^{i\pi J_x/\hbar}e^{i\pi J_y/\hbar}e^{i\pi J_z/\hbar} = 1.$$

Interpret this geometrically in terms of a sequence of rotations through π about each axis in turn.

3. Recall that we proved the Cauchy-Schwartz inequality, for any two vectors,

$$|\langle 1|2\rangle|^2 \leq \langle 1|1\rangle\langle 2|2\rangle. \tag{1}$$

Accept this as true; do not prove this. Suppose we have two Hermitian operators, x and p, that satisfy the commutation relation

$$[x, p] = i\hbar.$$

Suppose $|\ \rangle$ is some state in which the average value of x and p is $\bar{x}$ and $\bar{p}$, respectively,

$$\bar{x} = \langle\ |x|\ \rangle, \quad \bar{p} = \langle\ |p|\ \rangle.$$

(a) Show that

$$[x - \bar{x}, p - \bar{p}] = i\hbar.$$

(b) Show that

$$2\,\mathrm{Im}\,\langle\ |(x-\bar{x})(p-\bar{p})|\ \rangle = \hbar.$$

(c) Then establish from the Cauchy-Schwartz inequality (1) applied to the states $(x-\bar{x})|\ \rangle$ and $(p-\bar{p})|\ \rangle$ the inequality

$$\Delta x\,\Delta p \geq \frac{\hbar}{2},$$

where

$$(\Delta x)^2 = \langle\ |(x-\bar{x})^2|\ \rangle, \quad (\Delta p)^2 = \langle\ |(p-\bar{p})^2|\ \rangle.$$

This is the Heisenberg uncertainty relation.

4. In a state n, the mean value of a physical property is

$$\langle A\rangle_n = \langle n|A|n\rangle = \operatorname{tr}\rho A,$$

where $\rho = |n\rangle\langle n|$.

(a) Consider the state of a spin-1/2 system with $\sigma_z = +1$. The selective measurement symbol describing this state is

$$|+z,+z| = \frac{1+\sigma_z}{2}.$$

Compute $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, and $\langle\sigma_z\rangle$ in this state, using the algebraic properties of the σ's.

(b) Compute in the same state, with the angle ψ real,

$$\langle e^{i\sigma_{z'}\psi}\rangle$$

where the z' direction makes an angle θ with respect to the z direction, simply using the fact that $\sigma_{z'}^2 = 1$.

(c) Using the explicit representation

$$\sigma_{z'} = \sigma_x \sin\theta\cos\phi + \sigma_y \sin\theta\sin\phi + \sigma_z\cos\theta$$

show that $\sigma_{z'}^2 = 1$ by virtue of the algebraic properties of the σ operators.

(d) Derive the result of part 4b in another way, using

$$\langle +z|e^{i\sigma_{z'}\psi}|+z\rangle = \sum_{\sigma'_{z'},\sigma''_{z'}} \langle +z|\sigma'_{z'}\rangle\langle\sigma'_{z'}|e^{i\sigma_{z'}\psi}|\sigma''_{z'}\rangle\langle\sigma''_{z'}|+z\rangle.$$

Evaluate this using the wavefunctions

$$\psi_{+z'} = \begin{pmatrix}\langle +z|+z'\rangle\\ \langle -z|+z'\rangle\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix},$$

$$\psi_{-z'} = \begin{pmatrix}\langle +z|-z'\rangle\\ \langle -z|-z'\rangle\end{pmatrix} = \begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi/2}\\ \cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}.$$