Chapter 7

Wavefunctions

The transformation function $\langle a'|b'\rangle$ tells how to go from one description (a states) to another (b states)—from one class of states to another. But we don't have to think of all states in a class; we can talk of two states only:

$$b' \to 2, \quad a' \to 1. \tag{7.1}$$

Then the probability that if we have selected 2 of subsequently finding 1 (or *vice versa*) is

$$p(1,2) = |\langle 1|2\rangle|^2, \tag{7.2}$$

where $\langle 1|2\rangle$ is the (inner) product of the vector representing state 1 with the vector representing state 2. It tells how alike or different the two states are.

Now suppose we describe the system in terms of A measurements. We then use the algebraic construction of unity,

$$1 = \sum_{a'} |a'\rangle\langle a'|. \tag{7.3}$$

This carries us from vectors to components of vectors in some coordinate system:

$$\langle 1|2\rangle = \langle 1|\sum_{a'} |a'\rangle\langle a'||2\rangle = \sum_{a'}\langle 1|a'\rangle\langle a'|2\rangle. \tag{7.4}$$

To reiterate, since

$$|2\rangle = \sum_{a'} |a'\rangle\langle a'|2\rangle, \tag{7.5}$$

we can think of $|a'\rangle$ as a unit vector in some coordinate system, and $\langle a'|2\rangle$ as the component of the vector $|2\rangle$ in that coordinate system, the projection of $|2\rangle$ on the basis vector $|a'\rangle$. Since

$$\langle 1|2\rangle = \sum_{a'}\langle a'|1\rangle^*\langle a'|2\rangle, \tag{7.6}$$

we have here a complex scalar product, the sum of the products of components with complex conjugate components. The set of components are also denoted by

$$\langle a'|\,\rangle = \psi(a'), \tag{7.7}$$

where $|\,\rangle$ represents any state, which is the *wavefunction* of that state in the A description., Thus the probability of finding the state $|1\rangle$ given that the system was prepared in the state $|2\rangle$ is

$$p(1,2) = \left|\sum_{a'} \psi_1(a')^* \psi_2(a')\right|^2. \tag{7.8}$$

This is the form we saw much earlier when we discussed spin-1/2 system. See Sec. 4.2. Note that this probability makes no reference to A measurements, so is independent of the "coordinate system."

The probability that if the system is known to be in state 2, and measiurement of A will yield the value a' is

$$p(a',2) = |\langle a'|2\rangle|^2 = |\psi_2(a')|^2. \tag{7.9}$$

In general, if ψ is the wavefunction of the state in the A description, $|\psi(a')|^2$ is the probability of finding $A = a'$ in that state. Note the factorization referred to earlier:

- ψ refers to how the system is prepared,
- a' refers to what particular measurement is performed on that state.

The two probability statements (7.8) and (7.9) are not independent. For if

$$\langle 1| = \langle a'|, \quad |1\rangle = |a'\rangle, \tag{7.10}$$

then

$$\psi_1(a'') = \langle a''|a'\rangle = \delta(a'', a'), \tag{7.11}$$

and so Eq. (7.8) implies Eq. (7.9).

Finally, we note that the vector representing a state is a unit vector,

$$\langle 1|1\rangle = 1, \tag{7.12}$$

which physically expresses the fact that if we initially measure the system to be in state 1, it will be found with certainty in state 1 in a subsequent measurement. In terms of the a-wavefunctions,

$$1 = \langle 1|1\rangle = \sum_{a'} |\psi_1(a')|^2, \tag{7.13}$$

which says that the square of the length of a unit vector is the sum of the absolute squares of the components.

In summary, in mathematical language, we have gone just one step beyond Euclidean geometry to a *unitary geometry.* The space, the totality of vectors is not Euclidean, but what is called *Hilbert space.* In physical terms, we are describing a geometry of physical measurements—a geometry of states, and a state space.

7.1 Spin-1/2 example

We return to spin 1/2. We want to compute $\psi_{\sigma''_{z'}}(\sigma'_z)$, which is the component, with respect to spin measurments along the z direction, of a state with a certain spin value $(\sigma''_{z'})$ along the z' direction. The measurement symbols for these states are

$$|\sigma''_{z'} = +1| = \frac{1+\sigma_{z'}}{2}, \quad |\sigma''_{z'} = -1| = \frac{1-\sigma_{z'}}{2}. \tag{7.14}$$

We describe the system in terms of σ_z:

$$\sigma_{z'} = U^{-1}\sigma_z U, \tag{7.15}$$

by means of the unitary transformation

$$U = e^{i\frac{\theta}{2}\sigma_y} e^{i\frac{\phi}{2}\sigma_z}, \quad U^{-1} = e^{-i\frac{\phi}{2}\sigma_z} e^{-i\frac{\theta}{2}\sigma_y}. \tag{7.16}$$

Thus

$$\frac{1\pm\sigma_{z'}}{2} = U^{-1}\frac{1\pm\sigma_z}{2}U, \tag{7.17}$$

or

$$|\sigma''_{z'} = \pm 1| = U^{-1}|\sigma''_z = \pm 1|U. \tag{7.18}$$

Since the measurement symbols may be factored,

$$|\sigma''_{z'} = \pm 1| = |\sigma''_{z'}\rangle\langle\sigma''_{z'}|, \quad |\sigma''_z = \pm 1| = |\sigma''_z\rangle\langle\sigma''_z|, \tag{7.19}$$

we can factor statement (7.18) into

$$\langle\sigma''_{z'}| = \langle\sigma''_z|U, \quad |\sigma''_{z'}\rangle = U^{-1}|\sigma''_z\rangle. \tag{7.20}$$

Note that $\sigma''_{z'}$ and σ''_z represent the same outcome, the same number, but referring to measurements along different axes.

- All information about the direction of the axis of measurment is contained in U, relative to z, the standard direction.
- All information about the outcome of the measurement is contained in $|\sigma''_z\rangle$.

To work out the wavefunctions, the components of $|\sigma''_{z'}\rangle$ with respect to $|\sigma'_z\rangle$,

$$\psi_{\sigma''_{z'}}(\sigma'_z) = \langle\sigma'_z|\sigma''_{z'}\rangle = \langle\sigma'_z|U^{-1}|\sigma''_z\rangle, \tag{7.21}$$

we have to know the action of σ_x, σ_y, σ_z on $\langle\sigma'_z|$, $|\sigma'_z\rangle$. We recall

$$\sigma_x = |-+| + |+-| = |-\rangle\langle+| + |+\rangle\langle-|, \tag{7.22a}$$

$$\sigma_y = i|-+| - i|+-| = i|-\rangle\langle+| - i|+\rangle\langle-|, \tag{7.22b}$$

$$\sigma_z = |++| - |--| = |+\rangle\langle+| - |-\rangle\langle-|, \tag{7.22c}$$

$$1 = |++| + |--| = |+\rangle\langle+| + |-\rangle\langle-|. \tag{7.22d}$$

Therefore,

$$\begin{aligned}
\sigma_z|+\rangle &= |+\rangle, \quad \sigma_z|-\rangle = -|-\rangle, && (7.23a)\\
\sigma_x|+\rangle &= |-\rangle, \quad \sigma_x|-\rangle = |+\rangle, && (7.23b)\\
\sigma_y|+\rangle &= i|-\rangle, \quad \sigma_y|-\rangle = -i|+\rangle, && (7.23c)\\
1|+\rangle &= |+\rangle, \quad 1|-\rangle = |-\rangle. && (7.23d)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle+|\sigma_z &= \langle+|, \quad \langle-|\sigma_z = -\langle-|, && (7.24a)\\
\langle+|\sigma_x &= \langle-|, \quad \langle-|\sigma_x = \langle+|, && (7.24b)\\
\langle+|\sigma_y &= -i\langle-|, \quad \langle-|\sigma_y = i\langle+|, && (7.24c)\\
\langle+|1 &= \langle+|, \quad \langle-|1 = \langle-|. && (7.24d)
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\sigma''_{z'} &= +1\rangle \equiv |+,z'\rangle = U^{-1}|+,z\rangle\\
&= e^{-i\frac{\phi}{2}\sigma_z}\left(\cos\frac{\theta}{2} - i\sigma_y\sin\frac{\theta}{2}\right)|+\rangle\\
&= e^{-i\frac{\phi}{2}\sigma_z}\left(\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle\right)\\
&= e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}|+\rangle + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}|-\rangle\\
&= \sum_{\sigma'_z}|\sigma'_z\rangle\langle\sigma'_z|\sigma''_{z'} = +1\rangle = \psi_{+z'}(+)|+\rangle + \psi_{+z'}(-)|-\rangle, \qquad (7.25)
\end{aligned}$$

where $+z'$ means $\sigma''_{z'} = +1$, and $|+\rangle$ means $|+,z\rangle$. Thus we read off

$$\psi_{+z'}(+) = e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}, \quad \psi_{+z'}(-) = e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}. \qquad (7.26)$$

Similarly,

$$\begin{aligned}
|-,z'\rangle &= e^{-i\frac{\phi}{2}\sigma_z}\left(\cos\frac{\theta}{2} - i\sigma_y\sin\frac{\theta}{2}\right)|-\rangle = e^{-i\frac{\phi}{2}\sigma_z}\left(\cos\frac{\theta}{2}|-\rangle - \sin\frac{\theta}{2}|+\rangle\right)\\
&= -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}|+\rangle + e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}|-\rangle, \qquad (7.27)
\end{aligned}$$

from which we read off

$$\psi_{-z'}(+) = -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}, \quad \psi_{-z'}(-) = e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}. \qquad (7.28)$$

To get further exercise, work out the left vectors,

$$\begin{aligned}
\langle+,z'| &= \langle+|U = \langle+|\left(\cos\frac{\theta}{2} + i\sigma_y\sin\frac{\theta}{2}\right)e^{i\frac{\phi}{2}\sigma_z}\\
&= \left(\cos\frac{\theta}{2}\langle+| + \sin\frac{\theta}{2}\langle-|\right)e^{i\frac{\phi}{2}\sigma_z}\\
&= \langle+|\cos\frac{\theta}{2}e^{i\frac{\phi}{2}} + \langle-|\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}, \qquad (7.29)
\end{aligned}$$

which implies that

$$\psi_{+z'}(+)^* = \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}. \quad \psi_{+z'}(-)^* = \sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}. \tag{7.30}$$

which are indeed true, and also note

$$|\psi_{+z'}(+)|^2 + |\psi_{+z'}(-)|^2 = 1. \tag{7.31}$$

Similarly,

$$\begin{aligned}\langle -, z'| &= \langle -|U = \langle -|\left(\cos\frac{\theta}{2} + i\sigma_y \sin\frac{\theta}{2}\right)e^{i\frac{\phi}{2}\sigma_z} \\ &= \left(\cos\frac{\theta}{2}\langle -| - \sin\frac{\theta}{2}\langle +|\right)e^{i\frac{\phi}{2}\sigma_z} \\ &= \langle +|\left(-\sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\right) + \langle -|\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}},\end{aligned} \tag{7.32}$$

which correctly implies

$$\psi_{-z'}(+)^* = -\sin\frac{\theta}{2}e^{i\frac{\phi}{2}}, \quad \psi_{-z'}(-)^* = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}. \tag{7.33}$$

Finally, check that

$$\begin{aligned}0 &= \langle +z'| - z'\rangle = \psi_{+z'}(+)^*\psi_{-z'}(+) + \psi_{+z'}(-)^*\psi_{-z'}(-) \\ &= \left(e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\right)\left(-e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\right) + \left(e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\right)\left(e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\right) \\ &= -\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2} = 0.\end{aligned} \tag{7.34}$$

The $\pm$ signs, and the phases, which don't show unp in the probabilities

$$|\psi_{+z'}(+)|^2 = \cos^2\frac{\theta}{2}, \quad |\psi_{-z'}(+)|^2 = \sin^2\frac{\theta}{2}, \tag{7.35}$$

etc., are crucial for the above cancellation.

We can write the above wavefunction as column vectors,

$$\psi_{+z'} = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}, \quad \psi_{-z'} = \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}, \tag{7.36}$$

where the first row refers to the $\sigma_z = +1$ element, and the second row to $\sigma_z = -1$. These are nearly the same as the wavefunctions found earlier, in Eq. (4.35); they are equally as good as those (see homework).

Let's represent not just the states (vectors) but the algebraic symbols by arrays. In Eqs. (7.22a)–(7.22d), we express the symbols $\boldsymbol{\sigma}$, 1, in terms of the coefficients of the four measurement sybols $|\sigma_z'\rangle\langle z_z''|$:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{7.37a}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.37b}$$

where the rows represent the values of σ_z', the columns the values of σ_z''. This is an example of a more general procedure, which we sill describe in the next chapter.