Chapter 4

More About Spin

4.1 Higher Spins

We have been analyzing the results of the Stern-Gerlach experiment on the simplest possible atoms, those with spin 1/2. The next simplest situation occurs, for example, with O_2 molecules, where the beam of molecules from the oven gets split into three beams. See Fig. 4.1. The beam that is deflected up we interpret as having z-component of angular momentum $J_z = +\hbar$, that deflected down has $J_z = -\hbar$, and the undeflected has $J_z = 0$. We see that $\hbar$ is the universal scale of angular momentum differences. (The experiment measures μ_z, but since $\mu_z = \gamma J_z$, it likewise determines J_z.)

What is $\mathbf{J}$? The maximum values of J_z as our pictures suggest, which seem to imply that when J_z takes on its maximal value, $\mathbf{J}$ is aligned with the z axis, so that $J_x = J_y = 0$? No, this picture is incorrect. For

$$J^2 = J_x^2 + J_y^2 + J_z^2, \tag{4.1}$$

and if this is averaged over all the atoms in the beam

$$J^2 = \langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle = 3\langle J_z^2\rangle, \tag{4.2}$$

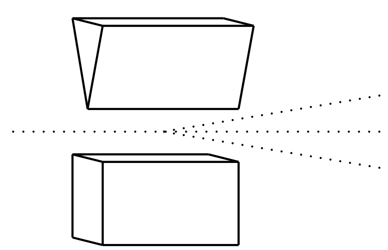

Figure 4.1: When an unpolartized beam of oxygen molecules enters a Stern-Gerlach apparatus, it gets split into three beams, one undeflected, one deflected up, and one deflected down.

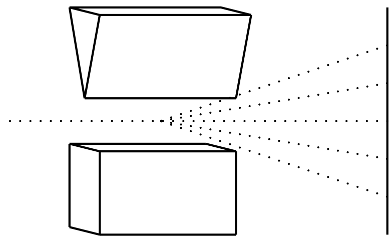

Figure 4.2: The general Stern-Gerlach experiment on atoms or molecules of spin j. The beam of unpolarized atoms is split into $n+1 = 2j+1$ beams of polarized atoms.

since the value of J^2 is fixed, and the distribution of spins is isotropic, so if we meansured any component of $\mathbf{J}$ we would get the same result on the average.

For the doublet (Ag),

$$J_z = \pm\frac{1}{2}\hbar, \quad J_z^2 = \frac{1}{4}\hbar^2, \tag{4.3}$$

so

$$J^2 = 3\langle J_z^2\rangle = \frac{3}{4}\hbar^2 > J_z^2, \tag{4.4}$$

simply because $J_x^2 = \frac{1}{4}\hbar^2$, $J_y^2 = \frac{1}{4}\hbar^2$. The naive anticipation is incorrect because

$$\langle J_x\rangle = 0 \quad \text{does not imply} \quad \langle J_x^2\rangle = 0. \tag{4.5}$$

We can measure J_z, but then we are unable to specify J_x, J_y—the angular momentum precesses uncontrollably about the z axis—there is a kind of uncertainty principle at work between the different components of $\mathbf{J}$. Similarly for the triplet (O_2),

$$J_z = (1, 0, -1)\hbar, \tag{4.6}$$

and each value is equally probable. Thus

$$\langle J_z^2\rangle = \frac{1}{3}\left[(\hbar)^2 + 0^2 + (-\hbar)^2\right] = \frac{2}{3}\hbar^2, \tag{4.7}$$

and therefore

$$J^2 = 3\langle J_z^2\rangle = 2\hbar^2 > (J_{z\,\max})^2 = \hbar^2. \tag{4.8}$$

Again, we cannot think of $\mathbf{J}$ as pointing along the z axis, when $J_z = \hbar$.

The general situation for the Stern-Gerlach experiment is illustrated in Fig. 4.2. The beam of atoms is split into $n+1$ subbeams. The one deflected the most upward has the maximal value of J_z,

$$J_{z\,\max} = j\hbar, \tag{4.9}$$

where j is called the angular momentum quantum number. What values can j assume? Because

$$j - (-j) = 2j = n, \tag{4.10}$$

j can either be an integer or an integer plus one-half:

$$j = \frac{1}{2}n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots. \tag{4.11}$$

For the doublet, $j = \frac{1}{2}$ (spin 1/2), for the triplet, $j = 1$ (spin 1). A spin 0 atom will go through the apparatus without being deflected.

Let's work out J^2 in general, for a spin-j atom. It is still true that

$$J^2 = 3\langle J_z^2\rangle, \tag{4.12}$$

where now

$$\langle J_z^2\rangle = \frac{1}{2j+1}\sum_{m=-j}^{j}(m\hbar)^2, \tag{4.13}$$

where m represents the possible (integer or integer plus 1/2) values $J_z/\hbar$ can assume; m is called the magnetic quantum number. We will show in the appendix that

$$\sum_{m=-j}^{j} m^2 = \frac{1}{3}(2j+1)j(j+1), \tag{4.14}$$

so

$$\langle J_z^2\rangle = \frac{1}{3}j(j+1)\hbar^2, \tag{4.15}$$

or

$$J^2 = j(j+1)\hbar^2. \tag{4.16}$$

This agrees with the special cases for $j = 1/2$ and $j = 1$ considered above. It is still true that

$$J^2 > J^2_{z\,\max}, \quad \text{because} \quad J^2 = j(j+1)\hbar^2, \quad J^2_{z\,\max} = j^2\hbar^2. \tag{4.17}$$

In fact, when $J_z = j\hbar$,

$$\frac{(J_x^2 + J_y^2)_{J_z=j\hbar}}{(J_{z\,\max})^2} = \frac{j}{j^2} = \frac{1}{j}, \tag{4.18}$$

which equals 2 for $j = 1/2$, but goes to zero as $j \to \infty$. The classical limit occurs when $j \to \infty$. Remember that the scale of angular momentum is set by $\hbar$, a tiny number for macroscopic physics, so macroscopic, classical, angular momenta correspond to very large values of j. For macroscopic angular momentum, the discreteness in J_z, J^2 becomes imperceptible. In the classical limit, indeed $J_{z\,\max}$ occurs when $\mathbf{J}$ points along the z axis.

4.2 State vectors

For spin-1/2 atoms, we have calculated the probability of finding $J_z = m\hbar$, $m = \pm 1/2$, given that the beam entering the Stern-Gerlach apparatus has $J_{z'} = m'\hbar$, $m' = \pm 1/2$:

$$p(m, m') = \begin{pmatrix} \cos^2\theta/2 & \sin^2\theta/2 \\ \sin^2\theta/2 & \cos^2\theta/2 \end{pmatrix}, \tag{4.19}$$

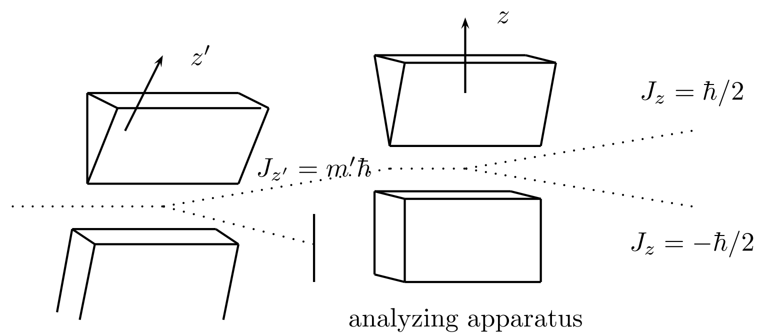

Figure 4.3: A preparing magnet prepares a state in which $J_{z'}$ has a definite value, and then an analyzing magnet measures J_z on those prepared atoms.

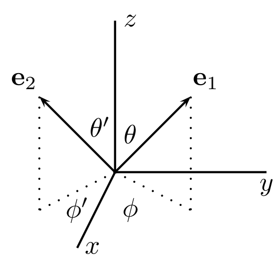

Figure 4.4: Spherical angles representing the directions of the analyzing and preparing magnets. Here θ, θ' are the polar angles, the angles $\mathbf{e}_1$ and $\mathbf{e}_2$ make with the z axis, while ϕ and ϕ' are the azimuthal angles the projection of $\mathbf{e}_1$ and $\mathbf{e}_2$ in the x-y plane make with the x axis.

where θ is the angle between the two directions along which the component of angular momentum was measured, z, z'. Two experiments are really involved—the "preparing" experiment, which selected atoms with $J_{z'} = m'\hbar$, and the "analyzing" experiment, which measured J_z on those atoms—See Fig. 4.3. We would like to separate the effects of these two measurements.

It helps to adopt a general coordinate system, as shown in Fig. 4.4. Here the figure shows the directions of the magnetic fields along the symmetry axis in the two magnets: $\mathbf{e}_1$ is the direction of $\mathbf{H}$ in the analyzing magnet, while $\mathbf{e}_2$ is the direction of $\mathbf{H}$ in the preparing magnet; both are unit vectors. Let the angle between the two directions be Θ,

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos\Theta. \tag{4.20}$$

On the other hand, in terms of Cartesian components,

$$\begin{aligned}\mathbf{e}_1\cdot\mathbf{e}_2 &= e_{1x}e_{2x}+e_{1y}e_{2y}+e_{1z}e_{2z}\\ &= \sin\theta\cos\phi\sin\theta'\cos\phi'+\sin\theta\sin\phi\sin\theta'\sin\phi'+\cos\theta\cos\theta',\end{aligned}\tag{4.21}$$

so

$$\cos\Theta=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi').\tag{4.22}$$

Now

$$p(+1/2,+1/2) = \cos^2\Theta/2=\frac{1+\cos\Theta}{2},\tag{4.23a}$$

$$p(-1/2,+1/2) = \sin^2\Theta/2=\frac{1-\cos\Theta}{2}.\tag{4.23b}$$

(4.23c)

It is convenient to work in terms of half-angles:

$$\begin{aligned}\cos\Theta &= \left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)\left(\cos^2\frac{\theta'}{2}-\sin^2\frac{\theta'}{2}\right)\\ &\quad+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}2\sin\frac{\theta'}{2}\cos\frac{\theta'}{2}\cos(\phi-\phi'),\end{aligned}\tag{4.24a}$$

while

$$1=\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)\left(\cos^2\frac{\theta'}{2}+\sin^2\frac{\theta'}{2}\right),\tag{4.24b}$$

so

$$\begin{aligned}\frac{1+\cos\Theta}{2} &= \cos^2\frac{\theta}{2}\cos^2\frac{\theta'}{2}+\sin^2\frac{\theta}{2}\sin^2\frac{\theta'}{2}\\ &\quad+2\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\cos(\phi-\phi').\end{aligned}\tag{4.25}$$

If $\phi-\phi'=0$, Eq. (4.25) is a square, of a product of two-component vectors:

$$\begin{aligned}\frac{1+\cos\Theta}{2} &= \left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}+\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^2\\ &= \left(\left(\cos\frac{\theta}{2},\sin\frac{\theta}{2}\right)\begin{pmatrix}\cos\frac{\theta'}{2}\\ \sin\frac{\theta'}{2}\end{pmatrix}\right)^2,\end{aligned}\tag{4.26}$$

where usual matrix row on column multiplication is understood. The first vector refers to the second experiment, the second to the first experiment. (As usual, we are reading formulas right to left.)

If $\phi-\phi'=\pi/2$, we get

$$\begin{aligned}\frac{1+\cos\Theta}{2} &= \left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\right)^2+\left(\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^2\\ &= \left|\cos\frac{\theta}{2}\cos\frac{\theta'}{2}+i\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right|^2.\end{aligned}\tag{4.27}$$

Here, we see the appearance of the absolute value of a complex number, written in term of the imaginary unit $i = \sqrt{-1}$. We see here a hint that the mathematical structure of quantum mechanics *requires* the use of complex numbers (in classical physics, complex numbers are only a convenience).

Because for complex numbers a and b

$$|a+b|^2 = (a+b)^*(a+b) = |a|^2 + |b|^2 + 2\mathrm{Re}\, a^* b, \tag{4.28}$$

and

$$\mathrm{Re}\, e^{-i(\phi-\phi')} = \cos(\phi - \phi'). \quad \left|e^{-i(\phi-\phi')}\right| = 1, \tag{4.29}$$

we have in general

$$\frac{1+\cos\Theta}{2} = \left|\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}e^{-i(\phi-\phi')}\right|^2. \tag{4.30}$$

Thus the probability is the absolute square of the product of two vectors:

$$p\left(\frac{1}{2},\frac{1}{2}\right) = \left|\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}e^{i\phi}\right)^* \begin{pmatrix} \cos\frac{\theta'}{2} \\ \sin\frac{\theta'}{2}e^{i\phi'} \end{pmatrix}\right|^2. \tag{4.31}$$

The two measurements are represented by vectors, which are here two component objects. The structure of each vector is the same, except that the first factor appears as

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}^\dagger = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}e^{-i\phi}\right), \tag{4.32}$$

where the † (adjoint symbol) means transposed, complex conjugated. The probability of going from one configuration, defined by the result of a measurement, to a second configuration, defined by the result of a second measurement, is obtained by complex multiplication of the vectors describing these two configurations, and taking the absolute square of the result.

If this is a consistent picture, this "factorization" must be possible for the other probability here:

$$\begin{aligned} p\left(-\frac{1}{2},\frac{1}{2}\right) &= \sin^2\frac{\Theta}{2} = \frac{1-\cos\Theta}{2} \\ &= \cos^2\frac{\theta}{2}\sin^2\frac{\theta'}{2} + \sin^2\frac{\theta}{2}\cos^2\frac{\theta'}{2} - 2\cos\frac{\theta}{2}\sin\frac{\theta'}{2}\sin\frac{\theta}{2}\cos\frac{\theta'}{2}\cos(\phi-\phi') \\ &= \left|-\sin\frac{\theta}{2}\cos\frac{\theta'}{2} + \cos\frac{\theta}{2}\sin\frac{\theta'}{2}e^{-i(\phi-\phi')}\right|^2, \end{aligned} \tag{4.33}$$

so this probability is also the absolute square of something. Can the quantity which is squared be factored? To do so, we note that $p(1/2, 1/2)$ is the probability of a transition from a state from $m' = 1/2$ along the z' axis (we call it a *state* because we know everything we can about it) to the state $m = 1/2$

along the z axis, while $p(-1/2,1/2)$ is the probability of a transition from a state from $m'=1/2$ along the z' axis to the state $m=-1/2$ along the z axis, These two situations have the same initial state in common. To see this in the mathematical description, we write

$$\begin{aligned} p\left(-\frac{1}{2},\frac{1}{2}\right) &= \left|-\sin\frac{\theta}{2}e^{i\phi}\cos\frac{\theta'}{2}+\cos\frac{\theta}{2}\sin\frac{\theta'}{2}e^{i\phi'}\right|^2 \\ &= \left|\left(-\sin\frac{\theta}{2}e^{-i\phi},\cos\frac{\theta}{2}\right)^*\begin{pmatrix}\cos\frac{\theta'}{2}\\ \sin\frac{\theta'}{2}e^{i\phi'}\end{pmatrix}\right|^2. \end{aligned} \tag{4.34}$$

The second factor, the same as before, represents the first measurement, which selects $m'=1/2$. The first factor represents the final measurement, which selects $m=-1/2$. The two vectors appearing in these probabilities are

$$\psi_{m=1/2}(\theta,\phi)=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix},\quad \psi_{m=-1/2}(\theta,\phi)=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}, \tag{4.35}$$

and the probability in all cases is

$$p(m,m')=\left|\psi_m(\theta,\phi)^\dagger\psi_{m'}(\theta',\phi')\right|^2. \tag{4.36}$$

In general, physical systems are represented by vectors in some space, a space which depends on how many options there are. The reason these vectors have only two components is that they represent systems with only two possible states, $J_z=\pm\hbar/2$. (Thus, to represent a system with 27 possible states, we would need a 27 component vector.) We see an intimation of an abstract geometry. A mathematics will be developed as a symbolic representation of experiments.

Before commencing on that, let us say a few more words about the vectors here, and their physical interpretation.

If we set $\theta=0$, that is, line up $\mathbf{e}_1$ with the z axis, we get

$$\psi_{1/2}=\begin{pmatrix}1\\0\end{pmatrix},\quad \psi_{-1/2}=\begin{pmatrix}0\\1\end{pmatrix} \tag{4.37}$$

$\psi_{1/2}$ represents a state in which $J_z=\frac{1}{2}\hbar$, while $\psi_{-1/2}$ represents a state in which $J_z=-\frac{1}{2}\hbar$. Then, if we first select a state with $J_{z'}=\frac{1}{2}\hbar$ and then measure J_z, where z' is oriented relative to z as shown in Fig. 4.5, the probability of finding $J_z=\frac{1}{2}\hbar$ is

$$\begin{aligned} p\left(\frac{1}{2},\frac{1}{2}\right) &= \left|(1,0)^*\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}\right|^2 \\ &= \left|\cos\frac{\theta}{2}\right|^2=\cos^2\frac{\theta}{2}, \end{aligned} \tag{4.38}$$

which is correct. In general, if $J_{z'}=m'\hbar$, $J_z=m\hbar$,

$$p(m,m')=\left|\psi_{m'}(m)\right|^2, \tag{4.39}$$

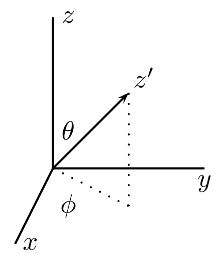

Figure 4.5: Orientation of z' relative to a coordinate system based on the z axis.

where the argument of the wavefunction is the component of the 2-vector. Using the wavefunctions (4.35), we reproduce the array of probabilities (4.19),

$$p(m, m') = \begin{pmatrix} \cos^2\theta/2 & \sin^2\theta/2 \\ \sin^2\theta/2 & \cos^2\theta/2 \end{pmatrix}. \tag{4.40}$$

The quantities ψ are called *probability amplitudes* or *wavefunctions*.

You take the absolute square of a probability amplitude to get a probability. The wave referred to is not classical, but one which mathematically gives the probability of finding the particle in a certain state.

We've been walking a tightrope between the classical and quantum worlds. Now we must begin again, and construct the quantum mechanics.

4.3 Appendix: Evaluation of $\sum_{m=-j}^{j} m^2$

A general procedure, which also gives sums of other powers of m, is as follows. Consider

$$S_j = \sum_{m=-j}^{j} e^{im\phi}. \tag{4.41}$$

Multiply this by $e^{i\phi}$:

$$e^{i\phi} S_j = \sum_{m=-j}^{j} e^{i(m+1)\phi} = \sum_{m=-j}^{j} e^{im\phi} + e^{i(j+1)\phi} - e^{-ij\phi}. \tag{4.42}$$

or

$$\begin{aligned} S_j\left(e^{i\phi} - 1\right) &= e^{i(j+1)\phi} - e^{-ij\phi} \\ &= e^{i\phi/2}\left(e^{i(j+1/2)\phi} - e^{-i(j+1/2)\phi}\right). \end{aligned} \tag{4.43}$$

That is,

$$S_j = \frac{e^{i(j+1/2)\phi} - e^{-i(j+1/2)\phi}}{e^{i\phi/2} - e^{-i\phi/2}} = \frac{\sin(j+1/2)\phi}{\sin\phi/2}. \tag{4.44}$$

From this general result, we can find the desired sum, by expanding in powers of ϕ:

$$\begin{aligned}
S_j &= \sum_{m=-j}^{j} \left(1 + im\phi - \frac{m^2\phi^2}{2} + \dots\right) \\
&= \frac{(j+1/2)\phi - \frac{1}{6}(j+1/2)^3\phi^3 + \dots}{\phi/2 - \frac{1}{6}(\phi/2)^3 + \dots} \\
&= 2j + 1 + \phi^2 \left[-\frac{1}{3}(j+1/2)^3 + \frac{1}{12}(j+1/2)\right] + \dots \\
&= 2j + 1 - \frac{1}{3}\phi^2 (j+1/2)j(j+1) + \dots . \qquad (4.45)
\end{aligned}$$

Thus we conclude:

$$\sum_{m=-j}^{j} 1 = 2j+1, \quad \sum_{m=-j}^{j} m = 0, \quad \sum_{m=-j}^{j} m^2 = \frac{1}{3}(2j+1)j(j+1). \qquad (4.46)$$