Chapter 11

Position and Momentum

As we have seen, a unitary operator maintains the lengths of all vectors. A simple example of a unitary operator consists in taking an orthonormal set of vectors,

$$\{\langle a'|\}, \quad \{|a'\rangle\}, \quad \langle a'|a''\rangle = \delta(a', a''), \tag{11.1}$$

and rearranging or relabeling them in some way. The new set of vectors, being collectively the same as the original set, is also orthonormal. This rearrangement then defines a unitary transformation. For example, suppose we have a system of n states, labeled by $a_1, a_2, \ldots, a_n$, that is $\{|a_k\rangle\}$. Define V as the transformation that replaces $\langle a_k|$ by $\langle a_{k+1}|$:

$$\langle a_k| \to \langle a_k|V = \langle a_{k+1}|. \tag{11.2}$$

Let us adopt a cylic notation so that

$$\langle a_{n+1}| = \langle a_1|, \quad \text{and} \quad \langle a_{n+k}| = \langle a_k|. \tag{11.3}$$

First, let's explicitly prove that V is unitary. From

$$\langle a_k|V = \langle a_{k+1}|, \tag{11.4}$$

by taking the adjoint we find

$$V^\dagger|a_l\rangle = |a_{l+1}\rangle, \tag{11.5}$$

so that

$$\langle a_k|VV^\dagger|a_l\rangle = \langle a_{k+1}|a_{l+1}\rangle = \delta_{kl} = \langle a_k|a_l\rangle, \tag{11.6}$$

for all k, l. Therefore, we conclude

$$VV^\dagger = 1. \tag{11.7}$$

Repeat the operation of V:

$$\langle a_k|V^2 = \langle a_{k+1}|V = \langle a_{k+2}|, \tag{11.8}$$

so repeating this n times,

$$\langle a_k|V^n = \langle a_{k+n}| = \langle a_k|, \tag{11.9}$$

we recover the original state for all k, so we conclude

$$V^n = 1. \tag{11.10}$$

This is a generalization of σ_z^2 to an n state system.

In the example of the unitary operator corresponding to a rotation of a spin-1/2 atom about the z axis,

$$e^{i\frac{\phi}{2}\sigma_z}, \quad \sigma_z^\dagger = \sigma_z. \tag{11.11}$$

As we have stated before, any unitary operator can be thought of as a function of a Hermitian operator,

$$U = e^{iH}, \quad U^\dagger = e^{-iH} = U^{-1}, \quad H^\dagger = H. \tag{11.12}$$

Since the concept of eigenvectors and eigenvalues applies to Hermitian operators, it also does to unitary operators.

Let us call the possible values of V, v', which must be solutions of the equation

$$(v')^n = 1, \tag{11.13}$$

which means that the v' are the n nth roots of unity,

$$v' = e^{2\pi ik/n}, \quad k = 0, 1, \dots, n-1, \tag{11.14}$$

or any equivalent choice, i.e., if n is odd,

$$k = -\frac{n-1}{2}, \dots, 0, \dots, \frac{n-1}{2}, \tag{11.15}$$

going through integer steps. Thus, for example, when $n = 3$,

$$k = -1, 0, 1, \quad \text{so} \quad v' = e^{-2\pi i/3},\, 1,\, e^{2\pi i/3}, \tag{11.16}$$

the three cube roots of 1. In general there are n distinct eigenvalues. The eigenvectors are automatically orthogonal and can be normalized to unity:

$$\langle v'|V = \langle v'|v', \quad V|v'\rangle = v'|v'\rangle. \tag{11.17}$$

These two equations are consistent with the adjoint operation, because taking the adjoint of the first equation gives

$$V^\dagger|v'\rangle = v'^*|v'\rangle, \tag{11.18}$$

which is the same as

$$V^{-1}|v'\rangle = v'^{-1}|v'\rangle, \tag{11.19}$$

which in turn implies

$$V|v'\rangle = v'|v'\rangle. \tag{11.20}$$

From this we can deduce

$$\langle v'|v''\rangle = \delta(v', v''). \tag{11.21}$$

The unit operator can be constructed as usual,

$$1 = \sum_{v'} |v'| = \sum_{v'} |v'\rangle\langle v'|. \tag{11.22}$$

To give an explicit construction of $|v'\rangle\langle v'|$ we recall that for any physical property (Hermitian operator)

$$\prod_{a'}(A - a') = 0, \quad |a'| = \prod_{a''\neq a'} \left(\frac{A - a'}{a' - a''}\right), \tag{11.23}$$

where the second equation says that, to construct $|a'|$ we simply remove a factor of $A - a'$ from the equation satisfied by A, apart for a constant factor. Here the equations satisfied by V, and by its eigenvalue v', are

$$V^n = 1, \quad (v')^n = 1, \tag{11.24}$$

so we can write

$$\left(\frac{V}{v'}\right)^n - 1 = 0 = \left(\frac{V}{v'} - 1\right)\left(\left(\frac{V}{v'}\right)^{n-1} + \left(\frac{V}{v'}\right)^{n-2} + \ldots + 1\right). \tag{11.25}$$

so we conclude that

$$|v'| = \text{constant} \sum_{l=0}^{n-1} \left(\frac{V}{v'}\right)^l. \tag{11.26}$$

We determine the constant by multiplying by $|v'\rangle$ on the right:

$$|v'\rangle = |v'\rangle\langle v'|v'\rangle = \text{constant} \sum_{l=0}^{n-1} \left(\frac{V}{v'}\right)^l |v'\rangle = \text{constant} \sum_{l=0}^{n-1} 1|v'\rangle, \tag{11.27}$$

where the sum is simply n. Therefore, the constant is $1/n$, and we have

$$|v'\rangle\langle v'| = \frac{1}{n} \sum_{l=0}^{n-1} \left(\frac{V}{v'}\right)^l. \tag{11.28}$$

Now multiply this construction by the last of the original vectors,

$$\langle a_n|v'\rangle\langle v'| = \frac{1}{n} \sum_{l=0}^{n-1} (v')^{-l} \langle a_n|V^l = \frac{1}{n} \sum_{l=0}^{n-1} (v')^{-l} \langle a_{l+n}|, \tag{11.29}$$

where the last left vector is $\langle a_l|$ because of the periodicity condition. Explicitly,

$$\langle a_n|v_k\rangle\langle v_k| = \frac{1}{n}\sum_{l=0}^{n-1} e^{-2\pi ikl/n}\langle a_l|. \tag{11.30}$$

Now if we multiply this on the right by $|a_n\rangle = |a_0\rangle$, we get

$$|\langle a_n|v_k\rangle|^2 = \frac{1}{n}, \tag{11.31}$$

which implies

$$\langle a_n|v_k\rangle = \frac{1}{\sqrt{n}}, \tag{11.32}$$

where we have chosen the phase to be zero. Thus we conclude from Eq. (11.30) that the two sets of vectors are related by a kind of Fourier series

$$\langle v_k| = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} e^{-2\pi ikl/n}\langle a_l|, \tag{11.33}$$

which implies that the general transformation function is

$$\langle v_k|a_l\rangle = \frac{1}{\sqrt{n}}e^{-2\pi ikl/n}. \tag{11.34}$$

The adjoint statement is

$$|v_k\rangle = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} e^{2\pi ikl/n}|a_l\rangle. \tag{11.35}$$

and

$$\langle a_l|v_k\rangle = \frac{1}{\sqrt{n}}e^{2\pi ikl/n}. \tag{11.36}$$

Note that the completeness statement

$$\sum_k \langle a_l|v_k\rangle\langle v_k|a_m\rangle = \frac{1}{n}\sum_k e^{2\pi ik(l-m)/n} = \delta_{lm}, \tag{11.37}$$

is the statement that

$$\sum_{k=0}^{n-1} e^{2\pi ikl/n} = 0, \quad l \neq 0, \quad l = 1, 2, \dots, n-1, \tag{11.38}$$

i.e.,

$$\sum_{k=0}^{n-1} (\mu_k)^l = 0, \quad l \neq 0, \quad l = 1, 2, \dots, n-1, \tag{11.39}$$

where μ_k is the kth nth root of unity, i.e., a solution to

$$(\mu_k)^n = 1. \tag{11.40}$$

Equation (11.39) says that

- the sum of the nth roots of unity is zero;
- the sum of the squares of the nth roots of unity is zero;
- $\ldots$,
- The sum of the $n-1$ powers of the nth roots of unity is zero.

Now beginning with the $|v_k\rangle$'s, we define another unitary operator by cyclic permutations:

$$U|v_k\rangle = |v_{k+1}\rangle, \tag{11.41}$$

so

$$U^n|v_k\rangle = |v_{k+n}\rangle = |v_k\rangle. \tag{11.42}$$

so

$$U^n = 1, \quad u' = e^{2\pi ik/n}, \quad k = 0, \ldots, n-1. \tag{11.43}$$

As above

$$|u'\rangle\langle u'| = \frac{1}{n}\sum_{l=0}^{n-1}\left(\frac{U}{u'}\right)^l, \tag{11.44}$$

and

$$|u_k\rangle\langle u_k|v_n\rangle = \frac{1}{n}\sum_{l=0}^{n-1} e^{-2\pi ikl/n}U^l|v_n\rangle, \tag{11.45}$$

where

$$U^l|v_n\rangle = |v_{n+l}\rangle = |v_l\rangle, \tag{11.46}$$

and

$$\langle v_n|u_k\rangle\langle u_k|v_n\rangle = |\langle u_k|v_n\rangle|^2 = \frac{1}{n}, \tag{11.47}$$

so making the simplest choice of phase,

$$|u_k\rangle = \frac{1}{\sqrt{n}}\sum_l e^{-2\pi ikl/n}|v_l\rangle = \sum_l |v_l\rangle\langle v_l|a_k\rangle = |a_k\rangle! \tag{11.48}$$

We are back to where we began. The whole closed system is

$$\langle u_k|V = \langle u_{k+1}|, \quad U|v_k\rangle = |v_{k+1}\rangle, \quad \langle u_k|v_l\rangle = \frac{1}{\sqrt{n}}e^{2\pi ikl/n}, \tag{11.49}$$

where the two sets of eigenvectors correspond to V and U, respectively,

$$\langle v_k|V = e^{2\pi ik/n}\langle v_k|, \quad U|u_k\rangle = e^{2\pi ik/n}|u_k\rangle. \tag{11.50}$$

We now need to consider the U and V operators together:

$$\langle u_k|VU = \langle u_{k+1}|U = \langle u_{k+1}|u_{k+1}, \quad u_{k+1} = u_k e^{2\pi i/n}, \tag{11.51a}$$

$$\langle u_k|UV = \langle u_k|u_k V = \langle u_{k+1}|u_k, \tag{11.51b}$$

or

$$\langle u_k|VU = \langle u_k|UVe^{2\pi i/n}, \tag{11.52}$$

for all k. Thus

$$VU = e^{2\pi i/n}UV. \tag{11.53}$$

More generally,

$$V^2U = e^{2\pi i/n}VUV = e^{2\pi 2i/n}UV^2, \tag{11.54}$$

etc., leads to

$$V^lU^k = e^{2\pi ikl/n}U^kV^l, \tag{11.55}$$

or, more symmetrically,

$$e^{\pi ikl/n}U^kV^l = e^{-\pi ikl/n}V^lU^k. \tag{11.56}$$

Note this is invariant under the substitution $U \to V$, $V \to U^{-1}$:

$$e^{\pi ikl/n}V^kU^{-l} = e^{-\pi ikl/n}U^{-l}V^k, \tag{11.57}$$

for pre- and post-multiplying this by U^l gives back Eq. (11.56) with k and l interchanged.

We are familiar with this system for $n = 2$, where

$$U^2 = V^2 = 1, \quad UV = -VU. \tag{11.58}$$

This is satisfied if

$$U = \sigma_x, \quad V = \sigma_y, \tag{11.59}$$

and the set of four operators is completed by

$$-iUV = \sigma_z. \tag{11.60}$$

Also,

$$\langle u_k|v_l\rangle = \frac{1}{\sqrt{n}}e^{2\pi ikl/n}, \quad \text{or} \quad |\langle u_k|v_l\rangle|^2 = \frac{1}{n}, \tag{11.61}$$

is familiar as a statement for spin

$$|\langle \sigma_x'|\sigma_y'\rangle|^2 = \frac{1}{2}, \tag{11.62}$$

since the two directions are perpendicular. For $n = 2$ there are four algebraic elements $|\pm\rangle\langle\pm|$, or alternatively and equivalently, the four operators 1, $\boldsymbol{\sigma}$. In general there are n^2 algebraic elements, and there are also n^2 U^kV^l's, $k = 0, 1, \ldots, n-1$, $l = 0, 1, \ldots, n-1$. Are these equivalent, alternative descriptions? Yes. The proof follows from our construction of the diagonal measurement symbol in Eq. (11.44),

$$|u_k\rangle\langle u_k| = \frac{1}{n}\sum_l e^{-2\pi ikl/n}U^l. \tag{11.63}$$

Then

$$|u_k\rangle\langle u_k|V^m = |u_k\rangle\langle u_{k+m}| = \frac{1}{n}\sum_l e^{-2\pi ikl/n}U^l V^m. \tag{11.64}$$

So any physical quantity of this system is a function of two fundamental quantities U and V. And they have the property that measurement of one removes all prior knowledge of the other:

$$|\langle u_k|v_l\rangle|^2 = \frac{1}{n}, \tag{11.65}$$

independent of k and l. Such pairs of physical quantities are called *complementary variables.*

It is interesting to analyze traces:

$$\begin{aligned}\frac{1}{n}\mathrm{tr}\, U^k V^l &= \frac{1}{n}\sum_{u'}\langle u'|U^k V^l|u'\rangle = \frac{1}{n}\sum_m (u_m)^k\langle u_{m+l}|u_m\rangle \\ &= \delta_{l0}\frac{1}{n}\sum_m (u_m)^k = \delta_{k0}\delta_{l0},\end{aligned} \tag{11.66}$$

that is, all $(1/n)U^k V^l$ have zero trace except for $k = l = 0$. This generalizes the zero trace of σ_x, σ_y, $-i\sigma_x\sigma_y = \sigma_z$, but not 1. So if we write an arbitrary physical quantity as

$$f(U, V) = \frac{1}{n}\sum_{kl} f_{kl}U^k V^l, \tag{11.67}$$

we have

$$\mathrm{tr}\, f(U, V) = f_{00}. \tag{11.68}$$

Alternatively, suppose we have a classical function of two variables,

$$f(u', v') = \frac{1}{n}\sum_{kl} f_{kl}(u')^k(v')^l, \tag{11.69}$$

so summing over the eigenvalues

$$\sum_{u'v'} f(u', v') = \frac{1}{n}\sum_{kl} f_{kl}\sum_{u'}(u')^k\sum_{v'}(v')^l = nf_{00}, \tag{11.70}$$

since the two sums are

$$\sum_{u'}(u')^k = n\delta_{n0}, \quad \sum_{v'}(v')^l = n\delta_{l0}. \tag{11.71}$$

Thus,

$$\mathrm{tr}\, f(U, V) = \frac{1}{n}\sum_{u'v'} f(u', v'), \tag{11.72}$$

where both sides represent a sum over all states.

We have a classification of different kinds of physical variables. As we see, this is very familiar for $n = 2$. How about the opposite limit, $n \to \infty$? We will find it convenient to make more explicit the Hermitian operator in $U = e^{iH}$, say. Let $n \to \infty$ through odd numbers so that [see Eq. (11.15)]

$$u' = e^{(2\pi i/n)k}, \quad k = \frac{n-1}{2}, \dots, 0, \dots, -\frac{n-1}{2}. \tag{11.73}$$

Define

$$\epsilon = \sqrt{\frac{2\pi}{n}}. \tag{11.74}$$

Then with q and p being Hermitian operators, defined by

$$U = e^{i\epsilon q}, \quad q' = \epsilon k, \quad \text{so} \quad u' = e^{i\epsilon^2 k} = e^{i2\pi k/n}, \tag{11.75a}$$
$$V = e^{i\epsilon p}, \quad p' = \epsilon l, \quad \text{so} \quad v' = e^{i\epsilon^2 l} = e^{i2\pi l/n}. \tag{11.75b}$$

Recalling from Eq. (11.55)

$$V^k U^l = e^{(2\pi i/n)kl} U^l V^k, \tag{11.76}$$

so that

$$e^{ik\epsilon p} e^{il\epsilon q} = e^{i\epsilon k\epsilon l} e^{il\epsilon q} e^{ik\epsilon p}, \tag{11.77}$$

or

$$e^{iq'p} e^{ip'q} = e^{iq'p'} e^{ip'q} e^{iq'p}, \tag{11.78}$$

or

$$e^{-iq'p} e^{ip'q} e^{iq'p} = e^{ip'(q-q')}. \tag{11.79}$$

But unitary transformation maintain algebraic relations, for example,

$$e^{-iq'p} f(q) e^{iq'p} = f(e^{-iq'p} q e^{iq'p}). \tag{11.80}$$

Evident here, power by power,

$$e^{iq'p}(q \cdots q) e^{iq'p} = e^{-iq'p} q e^{iq'p} e^{-iq'p} q e^{iq'p} e^{-iq'p} \cdots e^{-iq'p} q e^{iq'p}. \tag{11.81}$$

Therefore, as we saw in Eq. (11.79),

$$\exp\left[ip'\left(e^{-iq'p} q e^{iq'p}\right)\right] = e^{ip'(q-q')}. \tag{11.82}$$

As $\epsilon \to 0$, q', p' become continuous variables, and

$$e^{-iq'p} q e^{iq'p} = q - q', \tag{11.83}$$

and then, for arbitrarily small q',

$$(1 - iq'p) q (1 + iq'p) = q + iq'[q, p] = q - q', \quad [q, p] = qp - pq, \tag{11.84}$$

and therefore

$$[q, p] = i. \tag{11.85}$$

Alternatively, from Eq. (11.78),

$$e^{ip'q}e^{iq'p}e^{-ip'q} = e^{iq'(p-p')} = \exp\left[iq'\left(e^{ip'q}pe^{-ip'q}\right)\right], \tag{11.86}$$

implying

$$e^{ip'q}pe^{-ip'q} = p - p', \tag{11.87}$$

again implying

$$[q,p] = i. \tag{11.88}$$

Note the symmetry in this relation, $q \to p, p \to -q$.

Now consider the action on vectors,

$$\langle u_k|V^l = \langle u_{k+l}|, \tag{11.89}$$

or

$$\langle q'|e^{iq''p} = \langle q' + q''|, \tag{11.90}$$

where $q' = k\epsilon$, $q'' = l\epsilon$, and $q' + q'' = (k+l)\epsilon$. As $\epsilon \to 0$ we have continuous variation as $q'' \to 0$, and

$$e^{iq''p} \approx 1 + iq''p, \tag{11.91}$$

while

$$\langle q' + q''| \approx \langle q'| + q''\frac{\partial}{\partial q'}\langle q'|, \tag{11.92}$$

so

$$\langle q'|p = \frac{1}{i}\frac{\partial}{\partial q'}\langle q'|. \tag{11.93}$$

As a consistency check, consider

$$\begin{aligned}\langle q'|(qp - pq) &= q'\langle q'|p - \frac{1}{i}\frac{\partial}{\partial q'}\langle q'|q \\ &= q'\frac{1}{i}\frac{\partial}{\partial q'}\langle q'| - \frac{1}{i}\frac{\partial}{\partial q'}q'\langle q'| = \langle q'|i,\end{aligned} \tag{11.94}$$

which is consistent with Eq. (11.85). We see here an example of a general relation

$$\langle q'|f(q,p) = f\left(q', \frac{1}{i}\frac{\partial}{\partial q'}\right)\langle q'|. \tag{11.95}$$

If true for f_1 and f_2, it is true for $f_1 + f_2$ and for $f_1 f_2$:

$$\langle q'|\left(f_1(q,p) + f_2(q,p)\right) = \left(f_1\left(q', \frac{1}{i}\frac{\partial}{\partial q'}\right) + f_2\left(q', \frac{1}{i}\frac{\partial}{\partial q'}\right)\right)\langle q'|, \tag{11.96a}$$

$$\langle q'|f_1(q,p)f_2(q,p) = f_1\left(q', \frac{1}{i}\frac{\partial}{\partial q'}\right)f_2\left(q', \frac{1}{i}\frac{\partial}{\partial q'}\right)\langle q'|. \tag{11.96b}$$

This is certainly true for $f(q)$ and p, and therefore true for any algebraic combination of these ingredients.

Similarly,

$$U^k|v_l\rangle = |v_{l+k}\rangle, \tag{11.97}$$

or

$$e^{ip''q}|p'\rangle = |p'+p''\rangle, \tag{11.98}$$

where $p' = l\epsilon$, $p'' = k\epsilon$, and $p'+p'' = (l+k)\epsilon$. In the limit as $p'' \to 0$, we get

$$q|p'\rangle = \frac{1}{i}\frac{\partial}{\partial p'}|p'\rangle. \tag{11.99}$$

The adjoint statement is

$$\langle p'|q = i\frac{\partial}{\partial p'}\langle p'|. \tag{11.100}$$

Again we can check consistency with Eq. (11.85):

$$\langle p'|(qp - pq) = \left(i\frac{\partial}{\partial p'}p' - p'i\frac{\partial}{\partial p'}\right)\langle p'| = \langle p'|i. \tag{11.101}$$

Furthermore, we have the relation

$$\langle p'|f(q,p) = f(i\frac{\partial}{\partial p'}, p')\langle p'|, \tag{11.102}$$

at least for any algebraic combination of $f(p)$ and q.

11.1 Transformation functions and wavefunctions

Consider an arbitrary vector $|1\rangle$. Its u wavefunction may be written in terms of its v wavefunction by

$$\langle u'|1\rangle = \sum_{v'}\langle u'|v'\rangle\langle v'|1\rangle, \tag{11.103a}$$

or vice versa,

$$\langle v'|1\rangle = \sum_{u'}\langle v'|u'\rangle\langle u'|1\rangle, \tag{11.103b}$$

Thus the inner product between two state vectors is

$$\langle 1|2\rangle = \sum_{u'}\langle 1|u'\rangle\langle u'|2\rangle \tag{11.104a}$$

$$= \sum_{u'}\langle 1|v'\rangle\langle v'|2\rangle \tag{11.104b}$$

Define the q and p wavefunctions by

$$\langle u'|1\rangle = \sqrt{\epsilon}\psi_1(q'), \quad \langle v'|1\rangle = \sqrt{\epsilon}\psi_1(p'). \tag{11.105}$$

Then, in terms of these wavefunctions,

$$\langle 1|2\rangle = \sum_{q'}\epsilon\psi_1(q')^*\psi_2(q'), \tag{11.106}$$

so when we recognize that $q' = k\epsilon$, $dq' = (k+1)\epsilon - k\epsilon = \epsilon$, in the limit as $\epsilon \to 0$,

$$\langle 1|2\rangle = \int_{-\infty}^{\infty} dq' \psi_1(q')^* \psi_2(q'). \tag{11.107}$$

Alternatively,

$$\langle 1|2\rangle = \sum_{p'} \epsilon \psi_1(p')^* \psi_2(p') \to \int_{-\infty}^{\infty} dp' \psi_1(p')^* \psi_2(p'). \tag{11.108}$$

Also, since

$$\langle u'|v'\rangle = \frac{1}{\sqrt{n}} e^{2\pi ikl/n} = \frac{\epsilon}{\sqrt{2\pi}} e^{ik\epsilon l\epsilon} = \frac{\epsilon}{\sqrt{2\pi}} e^{iq'p'}, \tag{11.109}$$

we have

$$\psi(q') = \sum_{p'} \frac{\epsilon}{\sqrt{2\pi}} e^{iq'p'} \psi(p') \to \int_{-\infty}^{\infty} \frac{dp'}{\sqrt{2\pi}} e^{iq'p'} \psi(p'), \tag{11.110a}$$

and similarly

$$\psi(p') = \sum_{p'} \frac{\epsilon}{\sqrt{2\pi}} e^{-iq'p'} \psi(q') \to \int_{-\infty}^{\infty} \frac{dq'}{\sqrt{2\pi}} e^{-iq'p'} \psi(q'). \tag{11.110b}$$

We see that the momentum and position wavefunctions are related by Fourier transformation.

To find the physical interpretation of the wavefunction, we let $1 \to 2$ and

$$\langle\ |\ \rangle = 1 = \int_{-\infty}^{\infty} dq' |\psi(q')|^2 = \int_{-\infty}^{\infty} dp' |\psi(p')|^2, \tag{11.111}$$

and we infer that

- $dq'|\psi(q')|^2$ is the probability of finding q in the interval q' to $q' + dq'$,
- $dp'|\psi(p')|^2$ is the probability of finding p in the interval p' to $p' + dp'$.

We check this by noting

$$|\langle u'|\ \rangle|^2 = \epsilon|\psi(q')|^2 = dq'|\psi(q')|^2, \tag{11.112a}$$
$$|\langle v'|\ \rangle|^2 = \epsilon|\psi(p')|^2 = dp'|\psi(p')|^2. \tag{11.112b}$$