Physics 5583. Electrodynamics II.
Second Midterm Examination
Spring 2011

April 20, 2011

Instructions: This examination consists of three problems. If you get stuck on one part, assume a result and proceed onward. Do not hesitate to ask questions. GOOD LUCK!

1. This problem concerns the motion of an electron in a circular orbit of radius R under the influence of a constant magnetic field B on and inside the orbit.

   (a) Determine the relation between the angular frequency of the electron in its orbit, ω_0, and the magnetic field in terms of the charge and mass of the electron, e and m, in the nonrelativistic regime, $v \ll c$.

   (b) Using the dipole radiation formula,

   $$P = \frac{2}{3}\frac{\ddot{\mathbf{d}}^2}{4\pi c^3},$$

   where $\mathbf{d}$ is the electric dipole moment of the radiating system, compute the power radiated P in terms of e, ω_0, and R.

   (c) Relativistically, what is the relation between the energy of the circulating electron, E, and e, B, R, and v/c?

   (d) Relativistically, the power found in part 1b is enhanced by the factor $(E/mc^2)^4$, where now m is the rest mass of the electron. By equating P and $-dE/dt$, obtain a differential equation for dE/dt in terms of E only (apart from constants), that is, eliminate ω_0 and R, in the ultrarelativistic limit, $v/c \approx 1$.

(e) Assuming that this equation holds for all time, integrate it, and find an expression for the time required for the electron to radiate away all its energy, in terms of the initial energy E and the initial radius R. Evaluate the resulting expression assuming the initial energy of the electron is 1 GeV, the initial radius of the orbit is 1 m, and the electron rest energy is $mc^2 = 0.5$ MeV, and the express the answer in terms of fine structure constant,

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137},$$

where the unit conversion factor is $\hbar c = 2 \times 10^{-5}$ ev cm. An order of magnitude estimate is sufficient. Does the result agree with the formula obtained in class for the energy lost by an electron in one period,

$$\Delta E(\text{keV}) = 88.4\frac{E^4(\text{GeV})}{R(\text{m})}?$$

2. One can argue on the basis of relativistic invariance that the energy-momentum tensor for electromagnetism is given in terms of the field strength tensor $F^{\mu\nu}$ by

$$T_f^{\mu\nu} = F^{\mu\lambda}F^{\nu}{}_{\lambda} + kg^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta},$$

where k is some constant.

(a) Determine the constant k by the requirement that $T^{\mu\nu}$ be traceless.

(b) Then use Maxwell's equations in relativistic form in terms of $F^{\mu\nu}$ and the 4-vector current j^μ to compute $\partial_\mu T_f^{\mu\nu}$.

(c) The energy-momentum tensor for a charged particle of rest mass m is

$$T_p^{\mu\nu} = mc\int_{-\infty}^{\infty} d\tau\delta(x - x(\tau))\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}.$$

Use the relativistic equation of motion for the charged particle

$$m\frac{d^2x^\mu}{d\tau^2} = \frac{e}{c}F^{\mu\nu}\frac{dx_\nu}{d\tau}$$

to compute $\partial_\mu T_p^{\mu\nu}$, in terms of the current

$$\frac{1}{c} j^\mu(x) = \int_{-\infty}^{\infty} d\tau\, \delta(x - x(\tau)) e \frac{dx^\mu(\tau)}{d\tau}.$$

(d) What is $\partial_\mu (T_f^{\mu\nu} + T_p^{\mu\nu})$? Is this expected?

3. This problem concerns the Doppler effect.

   (a) The light wave emitted by a source is characterized by a four-vector propagation vector k^μ, with $k^\mu k_\mu = 0$. What is the relation between the frequency ω of the wave and the 3-vector wavevector $\mathbf{k}$?

   (b) Suppose in the rest frame of the source the wavevector has only an x component. Suppose an observer has a velocity relative to the source, v, in the same direction, x. Perform a Lorentz transformation to determine the frequency of the light as seen by the observer, ω_o, in terms of the frequency as seen by the source, ω. This could be called the longitudinal Doppler effect.

   (c) Alternatively, suppose the motion of the observer is in the perpendicular direction y, that is, $\mathbf{v}$ has only a y component. What is the frequency of the light as seen by the observer? This is sometimes called the transverse Doppler effect.

   (d) For low velocities, $v \ll c$, compare the magnitude of the transverse and the longitudinal Doppler frequency shifts.